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XIX.—*On a Formula representing the Mean Height of the Barometer at the Level of the Sea.* By Professor HANSTEEN of Christiania, in a Letter addressed to Professor FORBES, Secretary of the Royal Society of Edinburgh.

OBSERVATORY NEAR CHRISTIANIA, *26th September* 1846.

SIR,—You have communicated to me, that the Royal Society of Sciences in Edinburgh has done me the honour to elect me as a corresponding member. I beg you to render my humble thanks to the Society, and to assure, that it shall be my earnest wish to fulfil every task in my power which the Royal Society should demand.

That this letter may not reach your hands without any scientific communication, I subjoin the following :—From November 1822 to April 1824 inclusive, I observed the height of the barometer in Christiania, and found the mean reduced to 0° R., and to the level of the sea $=757^{m}\cdot763 = 335'''\cdot913$ lign. de Paris. As the mean height of the barometer observed at Paris by Bouvard, and reduced to 0°, and the level of the sea is $= 337'''\cdot53$, I was surprised at the great difference of $1'''\cdot62$ between Paris and Christiania. If p denotes the pressure of the atmosphere at the level of the sea, m and h the density of the mercury and its height in the tube, g the force of gravity, we have $p = m\,g\,h$, and, in another place, $p' = m\,g'\,h'$. If $p' = p$ is $g\,h = g'\,h'$, or $h' = \frac{g}{g'}\,h$. If, in the first place, the latitude is $= \phi$, in the second, $= \phi'$, we have $\frac{g}{g'} = \frac{1-0\cdot0025911 \cos 2\phi}{1-0\cdot0025911 \cos 2\phi'} = 1-0\cdot0025911\,(\cos 2\phi - \cos 2\phi')$; $h - h' = h,\ 0\cdot0025911\ (\cos\ 2\ \phi - \cos\ 2\ \phi')$. Taking $\phi = 0°$, $\phi' = 90°$, we have $h - h' = 1'''\cdot74$; and when $\phi = 48°\cdot50'$ (Paris), $\phi' = 59°\cdot55'$ (Christiania), we have $h - h' = 0'''\cdot32$. But the observations have given for Paris and Christiania $h - h' = 1'''\cdot62$; consequently, the mean pressure of the atmosphere is not the in different latitudes (" *Magazin for Naturvidensk.*" 1824, page 282–291).

Professor SCHOUW in Copenhagen has, in the Memoirs of the Royal Society of Sciences at Copenhagen for 1832 (page 291-342), collected all the known observations of the mean height of the barometer, which, with exactness, could be reduced to the level of the sea, and to 0° R. In the following table I have added the result of five years' observations here at the Observatory, and of the year 1844 at Bosekop. I have found that the observations can tolerably be represented bythe formula

$$\psi = 336'''\cdot8097 + 1'''\cdot3038 \cos 2\,\phi - 0'''\cdot7478 \cos 4\,\phi - 0'''\cdot9145 \cos 6\,\phi + 0'''\cdot5435 \cos 8\,\phi.$$

Place.	Observer.	Time.	φ	ψ Observed.	ψ Calculated.	Difference.
			° ′	‴	‴	‴
Christiansborg .	Trentepohl & Chenon	22 mo.	5 30	336·95	337·017	−0·067
Guayra	Boussingault . .	12 days	10 36	6·98	7·113	−0·131
St Thomas . . .	Hornbeck . . .	1 y.	18 19	7·13	7·497	−0·367
Rio Janeiro . .	Eschwege . . .	3 mo.	−22 54	8·69	7·876	+0·814
Santa Cruz, Teneriffe	Escolar	3 y.	28 28	8·77	8·360	+0·410
Madeira	Heinecken . . .	2 y.	32 36	9·20	8·635	+0·565
Cape of Good Hope	Puhlman and Wahlst	9 y.	−33 55	8·24	8·684	−0·444
Palermo . . .	Cacciatore . . .	35 y.	38 7	8·21	8·698	−0·488
Naples	Brioschi . . .	7 y.	40 51	7·94	8·554	−0·614
Florence . . .	Inghirami . . .	9 y.	43 47	7·76	8·262	−0·502
Avignon . . .	Guérin	10 y.	43 57	7·80	8·242	−0·442
Bologna . . .	Caturegli and Moratti	5 y.	44 30	7·87	8·170	−0·300
Padua	The Astronomers .	15 y.	45 24	7·87	8·044	−0·174
Paris	Bouvard . . .	11 y.	48 50	7·53	7·468	+0·062
London	Royal Society . .	7 y.	51 29	7·33	6·954	+0·376
Altona	Schumacher . .	6 y.	53 33	7·09	6·552	+0·438
Danzig	Strehlke . . .	2 y.	54 21	6·95	6·371	+0·579
Königsberg . .	Sommer	8 y.	54 43	6·95	6·297	+0·653
Apenrade . . .	Neuber	5 y.	55 3	6·72	6·231	+0·489
Edinburgh . . .	Forbes	3 y.	55 58	6·13	6·051	+0·079
Christiania . . .	Hansteen . . .	5 y.	59 55	6·18	5·362	+0·818
Reikiavik . . .	Thorslenson . .	12 y.	63 55	3·36	4·882	−1·522
Godthaab . . .	Mühlenpfort . .	5 y.	64 10	3·33	4·861	−1·531
Godhaven . . .	Graah and Fasting	2½ y.	68	4·19	4·694	−0·504
Bosekop . . .	Thomas	1 y.	69 58	5·39	4·715	+0·675
Melville Island .	Parry	1 y.	*74½	5·61	4·992	+0·618
Spitzbergen . .	Scoresby . . .	6–12 y.	75½	5·47	5·086	+0·384

The formula gives a *minimum* for $\phi = 0°$, and $\phi = 68°{\cdot}23'8$, and a *maximum* for $\phi = 36°{\cdot}12'6$, and $\phi = 90°$. The following table gives ψ for every fifth degree of latitude.

φ	ψ	φ	ψ
°	‴	°	°
0	336·995	45	338·101
5	7·012	50	7·246
10	7·096	55	6·240
15	7·291	60	5·345
20	7·623	65	4·801
25	8·057	70	4·715
30	8·478	75	5·037
35	8·714	80	5·561
40	8·612	85	6·034
45	8·101	90	6·216

* So in the original, and also in SCHOUW's Tables; but surely a mistake.—J. D. F.

The greatest observed difference is between Madeira and Godthaab, $339'''\cdot 2 - 333'''\cdot 3 = 5'''\cdot 9$; the formula gives only $4'''\cdot 0$. According to this table I have constructed a curve, which represents the variation of ψ from equator to the pole; but it is too voluminous for a letter.

If the observations are correct, there is a discontinuity between the eastern and western coasts of the Northern Atlantic Ocean; for instance, Christiania, Bosekop (Norway), compared with Reikiavik (Iceland), and Godthaab, Godhaven (Greenland); as also St Thomas (West Indies), compared with Santa Cruz, Teneriffe, and Madeira. If the whole globe was only an ocean, there would certainly be no such irregularities.

The Norwegian Government has, according to the demand of the Royal Society in London, resolved, that magnetical and meteorological observations, which stopped at the end of June 1843, shall be continued here a year. They were again commenced the 15th August this year, and will be continued to the same date in 1847. The unifilar is observed every tenth minute, mean time Göttingen; the bifilar and meteorological phenomena, every full hour. I am, Sir, sincerely yours,

CHRISTOPHER HANSTEEN.

[illegible]

[illegible]

[illegible]

[illegible]

XX.—*On General Differentiation. Part III.* By the Rev. P. KELLAND, *M.A., F.R.SS.L.&E., F.C.P.S., late Fellow of Queen's College, Cambridge; Professor of Mathematics, &c., in the University of Edinburgh.*

(Read December 21, 1846.)

Nearly six years ago, I presented to the Society two Memoirs on the subject of Differentiation, with fractional indices. The method which I adopted to extend the signification of a differential coefficient consisted in assuming that the function Γ, which enters into the value of the coefficient deduced from a particular hypothesis, is limited only by the definition $\Gamma(n+1)=n\Gamma(n)$. This generalization appears to be perfectly satisfactory, and promises to offer, if not the only, certainly the best extension of the Differential Calculus. Considering the length of the interval which has elapsed since the publication of my former Memoirs, it is remarkable that so little addition has been made to our knowledge of this branch of analysis. With the exception of one or two papers in LIOUVILLE's *Journal*, and a few remarks by Professor DE MORGAN, in his Treatise on the Differential Calculus (pp. 598–600), I am not aware that anything has been written on this subject since that time. Seeing, therefore, that others are not willing to enter on this very promising field, I consider it not improper that I should make known a number of extensions of this science to which I have been subsequently led, many of which have been in my possession a considerable time.

I must premise, that the object of this generalization of the differential calculus is, not only to extend the bounds of research beyond the limits of that science, but also to group and classify the results of the science itself. It is, perhaps, as important in the latter aspect as in the former; for its very first consequence is the union of the elementary forms of the two separate branches of that science—the differential and the integral calculus—into one, so that the integral becomes simply the negative differential. Now it is evident that this can only be done by extending to some form, which is general for the existing calculus, a universal and unrestrained interpretation. Such a form, properly selected, becomes, in the new science, a defining property, precisely in the same way that the common differential coefficient is the defining property of the differential calculus. There are several forms which might appear appropriate to this purpose: that which I have adopted is the differential coefficient of x^n. The assumption, therefore, on which the science is based, is the following: that

$\dfrac{d^\mu x^n}{d x^\mu} = (-1)^\mu \dfrac{\lfloor -n+\mu}{\lfloor -n} x^{n-\mu}$, whatever be n and μ. This form can be proved to be the correct one in every interpretable case, and can be deduced from the generalization of $\dfrac{d^\mu e^{cx}}{d x^\mu}$ when n is negative.* We shall at present assume it as the *defining property* or *definition* of $\dfrac{d^\mu x^n}{d x^\mu}$.

When, from this definition, we can deduce the differential coefficients of e^{cx} and of $\log x$, that is, of the ascending and descending index-function, we are in possession of the three fundamental forms from which all others may be derived. The following mode of arriving at those differential coefficients is different from that which has hitherto been given, and appears to leave nothing to be desired.

1. *To find* $\dfrac{d^\mu e^{cx}}{d x^\mu}$.

$$e^{cx} = 1 + cx + \frac{c^2 x^2}{1 \,.\, 2} + \frac{c^3 x^3}{1 \,.\, 2 \,.\, 3} + \text{\&c.} = \Sigma \frac{c^r x^r}{\lfloor r+1}.$$

$$\therefore \quad \frac{d^\mu e^{cx}}{d x^\mu} = (-1)^\mu \, \Sigma \frac{\lfloor -r+\mu}{\lfloor -r} \frac{c^r x^{r-\mu}}{\lfloor r+1}.$$

$$= (-c)^\mu \frac{\lfloor \mu}{\lfloor 0} \left\{ (c\,x)^{-\mu} + \frac{(c\,x)^{1-\mu}}{1-\mu} + \frac{(c\,x)^{2-\mu}}{(1-\mu)(2-\mu)} + \text{\&c.} \right\}$$

$$= (-c)^\mu \frac{\lfloor \mu}{\lfloor 0} \left\{ z^{-\mu} + \frac{d^{-1}}{d z^{-1}} z^{-\mu} + \frac{d^{-2}}{d z^{-2}} z^{-\mu} + \text{\&c.} \right\}, \text{ where } z = cx;$$

* See Part I., and the excellent Memoir of M. LIOUVILLE, referred to in that Treatise. Another formula has been proposed, viz.

$$\frac{d^\mu x^n}{d x^\mu} = \frac{\lfloor 1+n}{\lfloor 1+n-\mu} x^{n-\mu}.$$

I have lately received from Mr W. CENTER, of Langside, some judicious remarks on these formulæ, contrasting the results arrived at by them respectively. He shews that (without continual introduction of an infinite arbitrary constant) the latter formula is inapplicable in many of the most simple cases: for example, in d^μ of $\dfrac{1}{1+x}$ expanded positively, it gives, when applied, infinity on one side and not on the other, and when expanded negatively, infinity on both sides; and again, it gives for $\dfrac{d^\mu a}{d x^\mu}$ or $\dfrac{d^\mu a x^0}{d x^\mu}$ the value $\dfrac{1}{\lfloor 1-n} x^{-\mu}$, which is a function of x when μ is a positive proper fraction.

$$=(-c)^{\mu}\frac{\overline{|\mu}}{\overline{|0}}\left\{1-\left(\frac{d}{dz}\right)^{-1}\right\}^{-1}z^{-\mu}=c^{\mu}\left(\frac{d}{dz}-1\right)^{-1}\frac{d^{\mu+1}}{dz^{\mu+1}}\cdot 1$$

Let $$y=\left(\frac{d}{dz}-1\right)^{-1}\frac{d^{\mu+1}}{dz^{\mu+1}}\cdot 1\text{; then}$$

$$\frac{dy}{dz}-y=\frac{d^{\mu+1}}{dz^{\mu+1}}\cdot 1$$

$$y=e^{z}\left(C+\int dz\,e^{-z}\frac{d^{\mu+1}}{dz^{\mu+1}}\cdot 1\right)$$

Now $\frac{d^{\mu+1}}{dz^{\mu+1}}1=0$, except when μ is a negative whole number; in which case

$$\frac{d^{\mu+1}}{dz^{\mu+1}}1=\frac{z^{-\mu-1}}{\overline{|-\mu-1}}.$$

$\therefore\;\; y=Ce^{z}$; except when μ is a negative whole number, in which case

$$y=Ce^{z}-\frac{z^{-\mu-1}}{\overline{|\mu-2}}-\frac{z^{-\mu-1}}{\overline{|\mu-2}}-\&c.$$

Now, *in all cases* we omit the arbitrary functions in differentiation to any index; they being readily supplied when required. But $\frac{z^{-\mu-1}}{\overline{|-\mu-1}}$ + &c., is evidently included in the arbitrary function, in the case in question; we may therefore omit it, and write generally,

$$y=Ce^{z},\text{ or}$$

$$\frac{d^{\mu}e^{cx}}{dx^{\mu}}=c^{\mu}Ce^{z}=c^{\mu}Ce^{cx}\quad\ldots\ldots\quad(1)$$

This result has been deduced from the definition without any assumption whatever relative to the function Γ, except that it satisfies the condition $\overline{|n+1}=n\overline{|n}$. We may, consequently, obtain the value of the constant C, by admitting, that when n is positive, $\overline{|n}$ coincides with LEGENDRE's function Γ. In this case,

$$\frac{\overline{|n}}{x^{n}}=\int_{0}^{\infty}e^{-ax}a^{n-1}da.$$

Therefore, differentiating, to the index μ,

$\frac{\overline{|n+\mu}}{x^{n+\mu}}=C\int a^{\mu+n-1}e^{-ax}da$, by the definition and equation (1).

But if $n+\mu$ be positive, $\overline{|n+\mu}$ also coincides with LEGENDRE's function, therefore,

$$\frac{\overline{|n+\mu}}{x^{n+\mu}}=\int e^{-ax}a^{n+\mu-1}da,\text{ or }C=1.$$

Now C is altogether independent of n: if, therefore, we take n positive and greater than $(-\mu)$, which can always be done, we shall have proved generally, that

$$\frac{d^\mu e^{cx}}{d x^\mu} = c^\mu e^{cx} \dots (2).$$

It will be observed that the properties on which the truth of equation (2) is based, are these,—

$$\left.\begin{array}{ll} 1. & \dfrac{d^\mu x^n}{d x^\mu} = (-1)^\mu \dfrac{\Gamma(-n+\mu)}{\Gamma(-n)} x^{n-\mu} \\ 2. & \Gamma(n+1) = n\,\Gamma(n) \end{array}\right\} \text{whatever be } n.$$

$$3. \quad \frac{\Gamma(n)}{x^n} = \int_0^\infty e^{-\alpha x} \alpha^{n-1} d\alpha, \text{ when } n \text{ is positive.}$$

2. *To find* $\dfrac{d^\mu \log x}{d x^\mu}$.

In my previous Memoir, Art. 19, I obtained an expression for $\dfrac{d^\mu \log x}{d x^\mu}$, by assuming that $\int \frac{dx}{x} = \log x$; an assumption which owes its correctness to the admitted possibility of the introduction of an arbitrary constant of integration. Consequently, the conclusions at which I arrived can only be correct generally, by the aid of an arbitrary function of differentiation. Now, it is our object to avoid the use of such functions, and to obtain expressions for the general differential coefficient of all functions which shall be complete in themselves, so far as relates to the satisfaction of every law of combination to which they may be subjected. It becomes necessary, therefore, to reject the equation $\int \frac{dx}{x} = \log x$, and to substitute in its place some other function of x. The following process appears to be perfectly satisfactory.

The value of $\dfrac{x^p - x^q}{p}$ is $\dfrac{1 + p \log x + \&c. - 1 - q \log x - \&c.}{p}$

$$= \log x - \frac{q}{p} \log x + \mathrm{A} p + \&c.$$

If, therefore, q be of a higher order than p, such as p^2, it is manifest that $\dfrac{x^p - x^q}{p}$ will be a simple representation of $\log x$, provided $p=0$ and $\dfrac{q}{p}=0$.

By adopting this mode of representation we obtain,

$$\frac{d^\mu \log x}{d x^\mu} = (-1)^\mu \frac{\Gamma(\mu-p)}{p\,\Gamma(-p)} \frac{1}{x^{\mu-p}} - (-1)^\mu \frac{\Gamma(\mu-q)}{p\,\Gamma(-q)} \frac{1}{x^{\mu-q}}.$$

This expression comprehends every case, and appears to be the most simple

form under which the μth differential coefficient of a logarithm can be represented.

We shall reduce it in the different cases:

1. When μ is a negative whole number $=-m$.

$$\overline{|\mu-p}=\overline{|-(m+p)}\text{; and }\overline{|-p}=(-p-1)\overline{|-p-1}$$

$$=(-p-1)(-p-2)\ldots(-p-m)\overline{|-(m+p)}$$

$$=(-1)^m\frac{\overline{|1+m+p}}{\overline{|1+p}}\overline{|-(m+p)}$$

$$\therefore\quad \frac{\overline{|\mu-p}}{\overline{|-p}}=(-1)^\mu\frac{\overline{|1+p}}{\overline{|1-\mu+p}}$$

$$\text{and}\quad \frac{\overline{|\mu-q}}{\overline{|-q}}=(-1)^\mu\frac{\overline{|1+q}}{\overline{|1-\mu+q}}$$

Hence $$\frac{d^\mu\log x}{dx^\mu}=(-1)^{2\mu}\frac{\overline{|1+p}}{\overline{|1-\mu+p}}\,\frac{x^{-\mu+p}}{p}-(-1)^{2\mu}\frac{\overline{|1+q}}{\overline{|1-\mu+q}}\,\frac{x^{-\mu+q}}{p}$$

But $$\frac{\overline{|1+p}}{\overline{|1-\mu+p}}=\frac{1}{\overline{|1-\mu}}(1-p\,\mathrm{A}+\&c.)\text{ where }\mathrm{A}=\frac{1}{1}+\frac{1}{2}+\ldots\frac{1}{-\mu}$$

and $x^p=1+p\log x$ &c.

also $$\frac{\overline{|1+q}}{\overline{|1-\mu+q}}=\frac{1}{\overline{|1-\mu}}(1-q\,\mathrm{A}+\&c.)$$

$$x^q=1+q\log x+\&c.$$

$$\therefore\quad \frac{d^\mu\log x}{dx^\mu}=\frac{x^{-\mu}}{\overline{|1-\mu}}\cdot\frac{(1-p\,\mathrm{A}+\&c.)(1+p\log x+\&c.)}{p}$$

$$-\frac{x^{-\mu}}{\overline{|1-\mu}}\cdot\frac{(1-q\,\mathrm{A}+\&c.)(1+q\log x+\&c.)}{p}$$

$$=\frac{x^{-\mu}}{\overline{|1-\mu}}\left(\log x-\mathrm{A}-\frac{q}{p}\log x+\frac{q\,\mathrm{A}}{p}+\&c.\right)$$

$$=\frac{x^{-\mu}}{\overline{|1-\mu}}(\log x-\mathrm{A}),\text{ since }p\text{ and }\frac{q}{p}\text{ are both equal to }0.$$

Hence $\dfrac{d^{-m}\log x}{dx^{-m}}=\dfrac{x^m}{\overline{|m+1}}\left\{\log x-\left(\dfrac{1}{1}+\dfrac{1}{2}+\&c.+\dfrac{1}{m}\right)\right\}$ which is a well known expression for $\int^{(m)}dx^m\log x$

2. If μ be not a negative whole number, $\overline{|\mu}$ is finite; and

$$\frac{\overline{|\mu-p}}{p\overline{|-p}}=-\frac{\overline{|\mu-p}}{\overline{|1-p}}=-\overline{|\mu}(1+\mathrm{B}\,p+\&c.)$$

by supposing this function (which is finite) expanded in terms of p;

similarly
$$\frac{\overline{|\mu-q}}{q\overline{|-q}}=-\overline{|\mu}\,(1+\mathrm{B}q+\ \&c.)\,;$$

and from the expression in Art 2.

$$\begin{aligned}\frac{d^{\mu}\log x}{d\,x^{\mu}}&=(-1)^{\mu+1}\frac{\overline{|\mu}}{x^{\mu}}(1+\mathrm{B}\,p+\&c.)\,(1+p\log x+\&c.)\\&\quad-(-1)^{\mu+1}\frac{\overline{|\mu}}{x^{\mu}}(1+\mathrm{B}\,q+\&c.)\,(1+q\log x\ \&c.)\,\frac{q}{p}\\&=(-1)^{\mu+1}\frac{\overline{|\mu}}{x^{\mu}}.\end{aligned}$$

3. The expression given above for the differential coefficient of a logarithm is, therefore, perfectly general, and is applicable to all cases. It is essentially analytical in its nature, and does not appear to be reducible to a more arithmetical form so as to retain its general character. The expression which I previously gave exhibits very simply the nth differential coefficient of a logarithm as well as its nth integral, when n is a whole number, and may be, consequently, regarded as the most comprehensive arithmetical form of this function which we can at present obtain.

It may not be considered out of place here to introduce the deduction of the value of $\frac{d^n\log x}{d\,x^n}$, when n is a positive or a negative whole number, from this form also.

The equation is

$$\frac{d^n\log x}{d\,x^n}=\frac{\overline{|n}(-1)^{n+1}}{\overline{|-1}\,x^n}\left\{\log x-(n+1)n\left(\frac{1}{(n+1)n}+\frac{1}{n(n-1)}+\frac{1}{2}\,\frac{1}{(n-1)\,(n-2)}+\frac{1}{3}\,\frac{1}{(n-2)\,(n-3)}+\&c.\right)\right\}$$

(Part I, Art. 21.)

(1.) If n be a positive whole number, the only terms in this expression which are not indefinitely small, are,

$$\begin{aligned}&\frac{\overline{|n}\,(-1)^{n+2}}{\overline{|-1}\,x^n}(n+1)n\left(\frac{1}{n(n-n+1)\,(n-n)}+\frac{1}{(n+1)\,(n-n)\,(n-n-1)}\right)\\&=\frac{\overline{|n}\,(-1)^{n+2}}{\overline{|-1}\,x^n}\,(n+1)\,n\left(\frac{1}{n(n-n)}-\frac{1}{(n+1)\,(n-n)}\right)\\&=\frac{\overline{|n}\,(-1)^{n+2}}{\overline{|-1}\,x^n\,(n-n)}=\frac{\overline{|n}\,(-1)^{n+2}\,\overline{|n-n}}{x^n\,\overline{|n-n-1}\,\overline{|n-n+1}}\\&=\frac{\overline{|n}\,(-1)^{n+2}(n-n-1)}{x^n}=\frac{\overline{|n}\,(-1)^{n+3}}{x^n}=\frac{(-1)^{n+1}1\cdot2\ldots(n-1)}{x^n}\end{aligned}$$

the well known form.

(2.) If n be a negative integer $= -m$;

$$\text{Let } y=\frac{z^{m+1}}{n(n-1)}+\frac{1}{2}\frac{z^{m+2}}{(n-1)(n-2)}+\frac{1}{3}\frac{z^{m+3}}{(n-2)(n-3)}+\&c.$$

$$=\frac{z^{m+1}}{m(m+1)}+\frac{1}{2}\frac{z^{m+2}}{(m+1)(m+2)}+\frac{1}{3}\frac{z^{m+3}}{(m+2)(m+3)}+\&c.$$

$$\therefore \qquad \frac{d^2y}{dz^2}=z^{m-1}+\frac{1}{2}z^m+\&c.$$

$$=-z^{m-2}\log(1-z)$$

whence, by integration,

$$y=-\frac{z^m}{m(m-1)}\log(1-z)+\frac{1}{m(m-1)}\left\{\frac{z^m}{m}+\frac{z^{m-1}}{m-1}+\&c.+z\right\}$$

$$+\frac{1}{m(m-1)}\log(1-z)+\frac{1}{m-1}\left\{\frac{z^m}{m(m-1)}+\frac{z^{m-1}}{(m-1)(m-2)}+\&c.+\frac{z^2}{2\cdot 1}\right\}$$

$$+\frac{z}{m-1}\log(1-z)-\frac{z}{m-1}-\frac{1}{m-1}\log(1-z).$$

Consequently, the value of y between the limits 0 and 1 is

$$y=\frac{1}{m(m-1)}\left(\frac{1}{1}+\frac{1}{2}+\&c.\cdots+\frac{1}{m}\right)$$

$$+\frac{1}{m-1}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\&c.+\frac{1}{(m-1)m}\right)-\frac{1}{m-1}$$

$$=\frac{1}{m(m-1)}\left(\frac{1}{1}+\frac{1}{2}+\&c.+\frac{1}{m}\right)+\frac{1}{m}-\frac{1}{m-1}$$

$$=\frac{1}{m(m-1)}\left\{\frac{1}{2}+\frac{1}{3}+\&c.+\frac{1}{m}\right\}$$

and
$$\frac{d^{-m}\log x}{dx^{-m}}=\frac{\overline{|-m}(-1)^{-m+1}}{\overline{|-1}\,x^{-m}}\left\{\log x-1-m(m-1)y\right\}$$

$$=\frac{x^m(-1)^{-m+1}}{(-m)(-m+1)\ldots(-2)}\left\{\log x-\left(\frac{1}{1}+\frac{1}{2}+\&c.+\frac{1}{m}\right)\right\}$$

$$=\frac{x^m}{m(m-1)\ldots 2}\left\{\log x-\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\&c.+\frac{1}{m}\right)\right\}$$

which is the expression for $\int^{(m)} dx^m \log x$.

4. In my previous memoirs, I have obtained the general differential coefficients of several functions, and have applied the results to the solution of analytical and mechanical problems. It will be my object at present, to extend the science itself by exhibiting the solution of differential equations, and by investigating some of the properties of finite differences. In every instance I shall select the most simple problems which will serve to illustrate the process employed. Of the process itself, consisting entirely of the application of the *calculus of opera-*

tions, it is, perhaps, necessary to say a few words. The principle on which that calculus is founded is this:

If the laws which regulate the combinations of symbols of operation be the same as those which regulate the combinations of symbols of quantity, then all forms which would be equivalent relative to the latter, must also be equivalent relative to the former.

The *laws* to which symbols of quantity are subject, may be briefly classed under the seven following heads.

1. Their affections by numbers, or numerical quantities, are the same as if they themselves were numbers, or numerical quantities.
2. The law of signs.
3. The order of simple operations is indifferent.
4. The order of combined operations is indifferent.
5. Combined operations may be distributed.
6. and 7. The laws of indices.

Hence, if d, ϕ, ψ are any symbols of operation, subject to these laws (a and b being numerical quantities):

1. $(a \pm b)\,\phi = a\,\phi \pm b\,\phi = a\,\phi \pm \phi\,b$; &c.
2. $(a \pm \phi)\,(b \mp \psi) = a\,b \mp a\,\psi \pm b\,\phi - \phi\,\psi$; &c.
3. $\phi + \psi = \psi + \phi$
4. $\phi\,\psi = \psi\,\phi$
5. $d\,(\phi + \psi) = d\,\phi + d\,\psi$
6. $d^a\,d^b = d^{a+b}$
7. $(d^a)^b = d^{a\,b}$

results which would be equivalent were d, ϕ, ψ numerical quantities, are equivalent when they are operations. For example,

$$(d + \phi)^n = d^n + n d^{n-1}\,\phi + \frac{n\,(n-1)}{1\,.\,2}\,d^{n-2}\,\phi^2 + \text{\&c.}$$

The symbols of differentiation $\frac{d}{d\,x}$, $\frac{d}{d\,y}$ and of difference Δ_x, Δ_y satisfy these conditions.

It must be observed, in applying the principle which I have laid down, that it is inapplicable, unless it hold with respect to *every symbol* which enters into the operation. It will evidently apply to the ordinary symbols d and Δ as combined with each other, and to the symbols x, y as combined with each other; but it will not apply to the symbols d and x as combined with each other, because the fourth law is violated by their combination: For example,

$$d\,\Delta\,x^2 = 2, \quad \Delta\,d\,x^2 = 2$$

$$\therefore \quad d\,\Delta\,x^2 = \Delta\,d\,x^2:$$

But $\quad x\, d\, x^2 = 2\, x^2, \;\; d\, x\, x^2 = 3\, x^2$

$\therefore \quad x\, d\, x^2$ is not equal to $d\, x\, x^2$.

In proof of the sufficiency of the principle here laid down, it may be remarked, that both symbols of operation and symbols of quantity are defined or characterized by the above laws. The symbols of combination are indeed originally framed from arithmetic, but are subsequently generalized, and the basis of generalization is *obedience to these laws.* Thus the symbols + and − are generalized by *collective* symbols the reverse of each other, expressed by the equation $+a - a = +0 = -0$; where $+0$ is arithmetical, or signifies (as an operation *strictly*) *increased* by 0 : $\times$ and $\div$ are '*cumulative* symbols the reverse of each other,' expressed by the equation $\times a \div a = \times 1 = \div 1$; where $\times 1$ signifies strictly *multiplied* by 1. These definitions are in perfect conformity with the above laws. And a similar remarks applies to the general definition of an index.

Now certain symbols of operation, although not, like symbols of quantity, framed with direct reference to the above laws, do, notwithstanding, satisfy them. Consequently, *algebraic formulæ which are results of these laws and of nothing else, must be correct forms also when the algebraic symbols are replaced by such symbols of operation.*

Section I. Linear Differential Equations.

Preliminary Theorems.

5. Since $\left(\frac{d}{dx}\right)^{\mu} e^{cx} = c^{\mu} e^{cx}$, it is evident that if $f\left(\frac{d}{dx}\right)$ be any function whatever of $\frac{d}{dx}$, we shall have $f\left(\frac{d}{dx}\right) e^{cx} = f(c)\, e^{cx}$. (A).

Let u be a function of x, and suppose it expanded in the form $u = \Sigma\, a_m\, e^{mx}$; then

$$e^{rx}\, u = \Sigma\, a_m\, e^{(m+r)x}; \text{ and hence}$$

$$\left(\frac{d}{dx}\right)^{\mu} . \, e^{rx}\, u = \Sigma\, a_m\, (m+r)^{\mu}\, e^{(m+r)x}, \text{ by (A)}$$

$$= e^{rx}\, \Sigma\, a_m\, (m+r)^{\mu}\, e^{mx}$$

$$= e^{rx}\, \Sigma\, a_m \left(\frac{d}{dx} + r\right)^{\mu} e^{mx} \text{ by (A)}$$

$$= e^{rx} \left(\frac{d}{dx} + r\right)^{\mu} \Sigma\, a_m\, e^{mx}$$

$$= e^{rx} \left(\frac{d}{dx} + r\right)^{\mu} . \, u$$

$$\therefore \quad f\left(\frac{d}{dx}\right) . e^{rx} u = e^{rx} f\left(\frac{d}{dx}+r\right) . u \quad \text{(B)}$$

Let $x=e^{\theta}$, and suppose u expanded in the form $u=\Sigma a_n x^{-n}$: also write D for $\frac{d}{d\theta}$: then

$$x^{\mu} \left(\frac{d}{dx}\right)^{\mu} . \frac{1}{x^n} = (-1)^{\mu} \frac{\overline{|n+\mu}}{\overline{|n}} x^{-n}$$

$$\therefore \quad x^{\mu} \left(\frac{d}{dx}\right)^{\mu} . u = (-1)^{\mu} \Sigma a_n \frac{\overline{|n+\mu}}{\overline{|n}} x^{-n}$$

$$= (-1)^{\mu} \Sigma a_n \frac{\overline{|n+\mu}}{\overline{|n}} e^{-n\theta}$$

$$= (-1)^{\mu} \Sigma a_n \frac{\overline{|-\mathrm{D}+\mu}}{\overline{|-\mathrm{D}}} e^{-n\theta} \quad \text{by (A)}$$

$$= (-1)^{\mu} \frac{\overline{|-\mathrm{D}+\mu}}{\overline{|-\mathrm{D}}} \Sigma a_n x^{-n}$$

$$= (-1)^{\mu} \frac{\overline{|-\mathrm{D}+\mu}}{\overline{|-\mathrm{D}}} . u . \quad \text{(C)}$$

As a particular case of formula (B) we have

$$e^{r\theta} \frac{\overline{|-\mathrm{D}+\mu}}{\overline{|-\mathrm{D}}} . u = \frac{\overline{|-\mathrm{D}+\mu+r}}{\overline{|-\mathrm{D}+r}} . e^{r\theta} u . \quad \text{(D)}$$

These four theorems will be found of the utmost importance in reducing differential equations. Formulæ somewhat analogous have been applied to the solution of common differential equations by M. CAUCHY, *Exercices*, vol. i., p. 163, and *Exercices d'Analyse*, ii., 343; by Mr GREGORY, *Cambridge Mathematical Journal*, i., 22, &c.: and by Mr BOOLE, *Philosophical Transactions*, 1844, 225. Under the different heads in which we shall arrange differential equations, we shall solve only the most simple examples, our object being to illustrate the method of proceeding rather than to exhibit its power.

CLASS I. *Equations which are capable of solution without transformation.*

6. Ex. I. $\quad \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} - a^{\frac{1}{2}} y = 0.$

By writing d for $\frac{d}{dx}$, this equation becomes

$$(d^{\frac{1}{2}} - a^{\frac{1}{2}})\, y = 0 \quad \text{or } y = (d^{\frac{1}{2}} - a^{\frac{1}{2}})^{-1} . 0.$$

Suppose $y=\Sigma b_m e^{mx}$; then by (A)

$$\Sigma b_m (m^{\frac{1}{2}}-a^{\frac{1}{2}})\, e^{mx}=0\text{; which can only be satisfied when } m=a.$$

$\therefore \quad y=\mathrm{A}\,e^{ax}$ is the solution of the equation.

We might have proceeded in a somewhat different manner, as follows:

Put $0\,e^{mx}$ for 0, then

$$y=(d^{\frac{1}{2}}-a^{\frac{1}{2}})^{-1}\,.\,0\,e^{mx}=\frac{0\,e^{mx}}{m^{\frac{1}{2}}-a^{\frac{1}{2}}}\text{ by (A).}$$

But $\frac{0}{m^{\frac{1}{2}}-a^{\frac{1}{2}}}$ is finite only when $m=a$; and then it is constant; $\therefore\ y=\mathrm{A}\,e^{ax}$, as before.

Ex. 2. $\quad \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}-a^{\frac{1}{2}}y=\mathrm{X}$; X being any function of x.

We have $\quad y=(d^{\frac{1}{2}}-a^{\frac{1}{2}})^{-1}\,.\,\mathrm{X}+(d^{\frac{1}{2}}-a^{\frac{1}{2}})^{-1}\,.\,0.$

If $\quad \mathrm{X}=\Sigma\, b_r e^{rx}$

$$y=\mathrm{A}\,e^{ax}+\Sigma\,\frac{b_r}{r^{\frac{1}{2}}-a^{\frac{1}{2}}}\,e^{rx}\quad\text{(Ex. 1.)}$$

Cor. 1. If $r=a$, $\frac{b_r}{r^{\frac{1}{2}}-a^{\frac{1}{2}}}\,e^{rx}$ becomes infinite. In this case put $a+\alpha$ in place of r:

then $\frac{b_r}{r^{\frac{1}{2}}-a^{\frac{1}{2}}}\,e^{rx}$ becomes $b_r\,e^{ax}\,\dfrac{1+\alpha x+\&c.}{\dfrac{\alpha}{2\,a^{\frac{1}{2}}}+\&c.}$

$$=\frac{2\,a^{\frac{1}{2}}\,b_r}{\alpha}\,e^{ax}+2\,b_r\,a^{\frac{1}{2}}\,x\,e^{ax}\text{, when } \alpha=0;$$

of which the first term may be incorporated with $\mathrm{A}\,e^{ax}$; and the complete solution is

$$y=\mathrm{A}\,e^{ax}+2\,b_r\,a^{\frac{1}{2}}\,x\,e^{ax}+\Sigma\frac{b_s\,e^{sx}}{s^{\frac{1}{2}}-a^{\frac{1}{2}}}$$

Cor. 2. If $\mathrm{X}=b\,x^{-n}$, we have, by the well-known formula

$$\frac{1}{x^n}=\frac{1}{\Gamma n}\int_0^\infty e^{-\alpha x}\,\alpha^{n-1}\,d\alpha,$$

$$\therefore\quad (d^{\frac{1}{2}}-a^{\frac{1}{2}})^{-1}\,.\,\frac{1}{x^n}=\frac{1}{\Gamma n}\int_0^\infty\frac{e^{-\alpha x}\,\alpha^{n-1}d\alpha}{(-\alpha)^{\frac{1}{2}}-a^{\frac{1}{2}}}\text{ by (A.)}$$

$$=-\frac{1}{\Gamma n}\int_0^\infty\left(\frac{1}{a^{\frac{1}{2}}}+\frac{(-\alpha)^{\frac{1}{2}}}{a}+\frac{(-\alpha)}{a^{\frac{3}{2}}}+\&c.\right)e^{-\alpha x}\,\alpha^{n-1}\,d\alpha$$

$$=-\frac{1}{\overline{|n}}\left(\frac{\overline{|n}}{a^{\frac{1}{2}}x^{n}}+\frac{(-1)^{\frac{1}{2}}\overline{|n+\frac{1}{2}}}{a\,x^{n+\frac{1}{2}}}-\frac{\overline{|n+1}}{a^{\frac{3}{2}}x^{n+1}}-\text{\&c.}\right)$$

$$=-\frac{1}{a^{\frac{1}{2}}}\left(\frac{1}{x^{n}}-\frac{n}{a\,x^{n+1}}+\frac{n\,(n+1)}{a^{2}x^{n+2}}-\text{\&c.}\right)$$

$$-\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}\left(\frac{1}{a\,x^{n+\frac{1}{2}}}-\frac{n+\frac{1}{2}}{a^{2}x^{n+\frac{3}{2}}}+\frac{(n+\frac{1}{2})\,(n+\frac{3}{2})}{a^{3}x^{n+\frac{5}{2}}}+\text{\&c.}\right)$$

$$=a^{\frac{1}{2}}\,e^{a\,x}\int\frac{e^{-a\,x}}{x^{n}}\,d\,x+\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}\,e^{a\,x}\int\frac{e^{-a\,x}}{x^{n+\frac{1}{2}}}\,d\,x$$

$$\therefore\quad y=\mathrm{A}\,e^{a\,x}+b\,e^{a x}\left\{a^{\frac{1}{2}}\int\frac{e^{-a\,x}}{x^{n}}\,d\,x+\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}\int\frac{e^{-a\,x}}{x^{n+\frac{1}{2}}}\,d\,x\right\}$$

7. The solution of the foregoing examples might have been obtained very differently, thus:

If $\qquad d^{\frac{1}{2}}\,y-a^{\frac{1}{2}}\,y=\mathrm{X}\,;\ \ y=\frac{x}{d^{\frac{1}{2}}-a^{\frac{1}{2}}}=\frac{d^{\frac{1}{2}}+a^{\frac{1}{2}}}{d-a}\,.\,\mathrm{X}$

Now $\frac{1}{d-a}\mathrm{X}$ is the solution of the ordinary differential equation $\frac{d\,v}{d\,x}-a\,v=\mathrm{X}$; its value is, consequently, $e^{a\,x}\left(\int e^{-a x}\mathrm{X}\,d\,x+\mathrm{C}\right)$. Hence

$$y=\frac{d^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}}\,.\,e^{a\,x}\left(\,.\int e^{-a\,x}\,\mathrm{X}\,d\,x+\mathrm{C}\right)+a^{\frac{1}{2}}\,e^{a\,x}\int(e^{-a\,x}\,\mathrm{X}\,d\,x+\mathrm{C})$$

For instance, if $\mathrm{X}=0$, the solution of the equation is

$$y=2\,a^{\frac{1}{2}}\,\mathrm{C}\,e^{a\,x}\,;$$

which is the same as that given above.

8. Ex. 3. $\qquad \frac{d\,y}{d\,x}+\frac{a\,d^{\frac{1}{2}}y}{d\,x^{\frac{1}{2}}}+b\,y=0$

This may be written $(d+a\,d^{\frac{1}{2}}+b)\,.\,y=0$; or $(d^{\frac{1}{2}}-\alpha^{\frac{1}{2}})\,(d^{\frac{1}{2}}-\beta^{\frac{1}{2}})\,.\,y=0$; where $\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}=-a$, and $(\alpha\,\beta)^{\frac{1}{2}}=b$, or $\alpha^{\frac{1}{2}}$, $\beta^{\frac{1}{2}}$ are the roots of the equation $z^{2}+a\,z+b=0$.

$$\therefore\qquad y=\mathrm{A}(d^{\frac{1}{2}}-\alpha^{\frac{1}{2}})^{-1}\,.\,0+\mathrm{B}(d^{\frac{1}{2}}-\beta^{\frac{1}{2}})^{-1}\,.\,0$$

$$=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,e^{\beta x}\quad(\text{Ex. 1.})$$

Cor. 1. If $\alpha=\beta$, we must write $\alpha+e$ instead of β, and proceed as in similar cases.

The result is $\qquad y=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,x\,e^{\alpha x}$

Cor. 2. In precisely the same way we may find the solution of the equation

$$\frac{d^{\frac{3}{2}}y}{d\,x^{\frac{3}{2}}}+a\frac{d\,y}{d\,x}+b\,\frac{d^{\frac{1}{2}}y}{d\,x^{\frac{1}{2}}}+e\,y=0.$$

If $\alpha^{\frac{1}{2}}, \beta^{\frac{1}{2}}, \gamma^{\frac{1}{2}}$ be the roots of the equation $z^3 + a z^2 + b z + c = 0$, the solution is

$$y = \mathrm{A} e^{\alpha x} + \mathrm{B} e^{\beta x} + \mathrm{C} e^{\gamma x}$$

And a similar process applies to equations of all orders, with constant coefficients.

9. It will be seen that in solving these equations, we treat symbols of operation in exactly the same way as if they were symbols of quantity. Our justification for so doing is an appeal to the fact, that the laws which regulate the combination of the former symbols are precisely the same as those which regulate the combination of the latter. Were it otherwise,—were one of the symbols, for instance, to be subject to a different law relative to its combination with one class of symbols from that which regulates its combination with another, we should not be at liberty to separate the operations of such symbols, nor even to combine them otherwise than in the form in which they are actually presented to us. An example will illustrate this remark. The combination $(d^m d^n) \times (d^m d^n) \,.\, u$ may be written $(d^m \times d^n)^2 \,.\, u$, in which form it is equivalent to $d^{2m} d^{2n} \,.\, u$: but the combination $(d^m x^n) \times (d^m x^n) \,.\, u$, when written (as we shall write it) $(d^m x^n)^2 \,.\, u$, is not equivalent to $d^{2m} x^{2n} \,.\, u$. The commutative law, or the law according to which operations may be taken *in any order*, is not true of the symbols d^m, x^n, in their combination with one another.

We may remark, in addition, that when an operation on y has been changed into the reciprocal operation on 0 or on X, giving the solution $y = \frac{1}{(\mathrm{D}^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\mathrm{D}^{\frac{1}{2}} - \beta^{\frac{1}{2}})} 0$, for instance; the operation $\frac{1}{(\mathrm{D}^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(\mathrm{D}^{\frac{1}{2}} - \beta^{\frac{1}{2}})}$ is resolved into the two operations $\frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{1}{\mathrm{D}^{\frac{1}{2}} - \alpha^{\frac{1}{2}}} - \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{1}{\mathrm{D}^{\frac{1}{2}} - \beta^{\frac{1}{2}}}$, in the same manner as a fraction is resolved into its equivalent partial fractions. On this subject the reader may consult an excellent paper by Mr Boole, in the Cambridge Mathematical Journal, vol. ii., p. 114, where this method is first employed.

10. Ex. 4.
$$\frac{dy}{dx} + a \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + by = \mathrm{X}.$$

This gives
$$y = (d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^{-1} (d^{\frac{1}{2}} - \beta^{\frac{1}{2}})^{-1} \,.\, (\mathrm{X} + 0)$$

Now
$$\frac{\mathrm{X}}{(d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})(d^{\frac{1}{2}} - \beta^{\frac{1}{2}})} = \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{\mathrm{X}}{d^{\frac{1}{2}} - \alpha^{\frac{1}{2}}} - \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \frac{\mathrm{X}}{d^{\frac{1}{2}} - \beta^{\frac{1}{2}}}$$

$$\therefore \quad y = \mathrm{A} e^{\alpha x} + \mathrm{B} e^{\beta x} + \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left\{ (d^{\frac{1}{2}} - \alpha^{\frac{1}{2}})^{-1} \mathrm{X} - (d^{\frac{1}{2}} - \beta^{\frac{1}{2}})^{-1} \mathrm{X} \right\} \quad \text{(Ex. 3.)}$$

Cor. 1. If
$$\mathrm{X} = \Sigma\, b_r e^{rx};$$

$$y = \mathrm{A} e^{\alpha x} + \mathrm{B} e^{\beta x} + \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \Sigma\, b_r e^{rx} \left(\frac{1}{r^{\frac{1}{2}} - \alpha^{\frac{1}{2}}} - \frac{1}{r^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \right)$$

$$=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,e^{\beta x}+\Sigma\frac{b_r e^{rx}}{r+a\,r^{\frac{1}{2}}+b}$$

Cor. 2. If $\mathrm{X}=b_r$ a constant, $\therefore r=0$ and

$$y=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,e^{\beta x}+\frac{b_r}{b}$$

Cor. 3. If $r+a\,r^{\frac{1}{2}}+b=0$, r must be equal either to α or to β. Suppose $r=\alpha$; then $b_r e^{rx}\dfrac{1}{r+a\,r^{\frac{1}{2}}+b}$ becomes, by writing $\alpha^{\frac{1}{2}}+c$ in place of $r^{\frac{1}{2}}$,

$$b_r\frac{e^{\alpha x}(1+2\,\alpha^{\frac{1}{2}}c\,x+\&c.)}{2\,\alpha^{\frac{1}{2}}c+a\,c}=\mathrm{C}\,e^{\alpha x}+\frac{b_r x\,e^{\alpha x}2\,\alpha^{\frac{1}{2}}}{2\,\alpha^{\frac{1}{2}}+a}$$

$$\text{and } y=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,e^{\beta x}+b_r\frac{2\,\alpha^{\frac{1}{2}}x\,e^{\alpha x}}{2\,\alpha^{\frac{1}{2}}+a}+\Sigma\,b_s\frac{e^{sx}}{s+a\,s^{\frac{1}{2}}+b}$$

Ex. 5.
$$\frac{d^{-1}y}{d\,x^{-1}}+a\frac{d^{-\frac{1}{2}}y}{d\,x^{-\frac{1}{2}}}+b\,y=\mathrm{X}.$$

This gives
$$(d^{-1}+a\,d^{-\frac{1}{2}}+b)\,.\,y=\mathrm{X}.$$

or $(d^{-\frac{1}{2}}-\alpha^{-\frac{1}{2}})(d^{-\frac{1}{2}}-\beta^{-\frac{1}{2}})\,.\,y=\mathrm{X}$; where $\alpha^{-\frac{1}{2}}$, $\beta^{-\frac{1}{2}}$ are the roots of the equation $z^2+a\,z+b=0$;

$$\therefore\quad y=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,e^{\beta x}+\frac{(d^{-\frac{1}{2}}-\alpha^{-\frac{1}{2}})^{-1}\,.\,\mathrm{X}}{\alpha^{-\frac{1}{2}}-\beta^{-\frac{1}{2}}}-\frac{(d^{-\frac{1}{2}}-\beta^{-\frac{1}{2}})^{-1}\,.\,\mathrm{X}}{\alpha^{-\frac{1}{2}}-\beta^{-\frac{1}{2}}}$$

$$=\mathrm{A}\,e^{\alpha x}+\mathrm{B}\,e^{\beta x}+\frac{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}-\beta^{\frac{1}{2}}}\left\{(d^{\frac{1}{2}}-\alpha^{\frac{1}{2}})^{-1}\,.\,\alpha^{\frac{1}{2}}\frac{d^{\frac{1}{2}}\mathrm{X}}{d\,x^{\frac{1}{2}}}-(d^{\frac{1}{2}}-\beta^{\frac{1}{2}})^{-1}\beta^{\frac{1}{2}}\frac{d^{\frac{1}{2}}\,\mathrm{X}}{d\,x^{\frac{1}{2}}}\right\}$$

which is reduced to Ex. 2.

In precisely the same manner we may solve the more general equation $\dfrac{d^n y}{dx^n}$

$$+\,a\frac{d^{n-\alpha}y}{d\,x^{n-\alpha}}+b\,\frac{d^{n-2\alpha}\,y}{d\,x^{n-2\alpha}}+\&\text{c.}+y=\mathrm{X},\ n \text{ being a multiple of } \alpha.$$

Class II. *Elementary Equations.*

11. The form to which more complicated equations can generally be reduced is $y-m\,x^n\dfrac{d^{\frac{1}{2}}\,y}{d\,x^{\frac{1}{2}}}=\mathrm{X}$; and it is with equations of this form that we are now to be occupied. The simplest case, when $n=0$, we have already solved.

Ex. 1.
$$y-m\sqrt{x}\frac{d^{\frac{1}{2}}\,y}{d\,x^{\frac{1}{2}}}=0.$$

By (C) this is reduced to $y-m\sqrt{-1}\dfrac{\lceil -\mathrm{D}+\frac{1}{2}}{\lceil -\mathrm{D}}\,.\,y=0,$

or
$$y=\left(1-m\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\frac{1}{2}}}{\overline{|-\mathrm{D}}}\right)^{-1}.\,0.$$

Suppose
$$y=\Sigma\, a_n\, e^{-n\theta};\ \text{then}$$

$$\Sigma\, a_n\left(e^{n-\theta}-m\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}\,e^{-n\theta}\right)=0\ \text{by (A)};$$

which can be satisfied only by making $1-m\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}=0$; giving, consequently, only one value of n:

Hence
$$y=\frac{\mathrm{A}}{x^n}\ \text{is the complete solution.}$$

Cor. If
$$m=\frac{2}{\sqrt{-1}\sqrt{\pi}}=\frac{2}{\sqrt{-1}\,\overline{|\frac{1}{2}}}$$

$$\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}=\tfrac{1}{2}\,\overline{|\tfrac{1}{2}}=\overline{|\tfrac{3}{2}}=\frac{\overline{|1+\frac{1}{2}}}{\overline{|1}};$$

$$\therefore\qquad n=1\ \text{and}\ y=\frac{\mathrm{A}}{x}.$$

Ex. 2.
$$y-m\sqrt{x}\,\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=\mathrm{X}.$$

The equation in θ is $y-m\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\frac{1}{2}}}{\overline{|-\mathrm{D}}}\,y=\mathrm{F}\,e^{-\theta}$

$$\therefore\qquad y=\left(1-m\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\frac{1}{2}}}{\overline{|-\mathrm{D}}}\right)^{-1}.\,0+\left(1-m\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\frac{1}{2}}}{\overline{|-\mathrm{D}}}\right)^{-1}.\,\mathrm{F}\,e^{-\theta}$$

$$=\frac{\mathrm{A}}{x^n}+\Sigma\, b_r\left(1-m\sqrt{-1}\,\frac{\overline{|r+\frac{1}{2}}}{\overline{|r}}\right)^{-1}e^{-r\theta}\ \text{(Ex. 1 and A)}$$

Cor. If $r=n$; this expression becomes infinite. We must, in this case, write $n+c$ in place of r, expand in terms of c, and finally put $c=0$.

We have, thus,
$$\frac{e^{-r\theta}}{1-m\sqrt{-1}\,\frac{\overline{|r+\frac{1}{2}}}{\overline{|r}}}=\frac{e^{-n\theta}(1-c\,\theta+\&c.)}{1-m\sqrt{-1}\left(\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}+\frac{d}{dn}\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}.\,c+\&c.\right)}$$

$$=\frac{e^{-n\theta}}{1-m\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}}+\frac{c\,e^{-n\theta}\,\theta}{c\,m\sqrt{-1}\,\frac{d}{dn}\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}}+\&c.$$

$$=\frac{e^{-n\theta}}{1-m\sqrt{-1}\,\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}}+\frac{e^{-n\theta}\,\theta}{m\sqrt{-1}\,\frac{d}{dn}\frac{\overline{|n+\frac{1}{2}}}{\overline{|n}}}$$

$$\therefore \qquad y=\frac{\mathrm{A}}{x^n}+\frac{b_r}{m\sqrt{-1}}\,\frac{\log x}{x^n}\,\frac{1}{\dfrac{d}{dn}\dfrac{\overline{|n+\frac{1}{2}}}{\overline{|n}}}$$

$$+\Sigma\frac{b_s}{x^s}\,\frac{1}{1-m\sqrt{-1}\,\dfrac{\overline{|s+\frac{1}{2}}}{\overline{|s}}}$$

Ex. 3. $$y-a\sqrt{-1}\,x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0.$$

Suppose $$y=\Sigma a_r x^{-r};\ \text{then}$$

$$\Sigma\left(a_r x^{-r}+a\frac{\overline{|r+1}}{\overline{|r+\frac{1}{2}}}\,a_{r+\frac{1}{2}}\,x^{-r}\right)=0$$

or $$a_{r+\frac{1}{2}}=-\frac{1}{a}\,\frac{\overline{|r+\frac{1}{2}}}{\overline{|r+1}}\,a_r\ .\quad (1.)$$

Hence the lowest value of r is 0, and the values succeed at intervals of $\frac{1}{2}$.

$\therefore\ y=\mathrm{A}+\frac{\mathrm{A}_1}{\sqrt{x}}+\frac{\mathrm{A}_2}{x}+$ &c., with the relation expressed by (1). By substitution

$$\mathrm{A}_1=-\frac{1}{a}\frac{\overline{|\frac{1}{2}}}{\overline{|1}}\mathrm{A},\ \mathrm{A}_2=-\frac{1}{a}\frac{\overline{|1}}{\overline{|\frac{3}{2}}}\mathrm{A}_1,\ \mathrm{A}_3=-\frac{1}{a}\frac{\overline{|\frac{3}{2}}}{\overline{|2}}\ \text{\&c.}$$

$$\mathrm{A}_1=-\frac{1}{a}\sqrt{\pi}\,\mathrm{A};\ \mathrm{A}_2=\frac{1}{a^2}\frac{\overline{|1}\,\overline{|\frac{1}{2}}}{\overline{|\frac{3}{2}}\,\overline{|1}}\mathrm{A}=\frac{2}{a^2}\mathrm{A}$$

$$\mathrm{A}_3=-\frac{1}{a^3}\frac{\overline{|\frac{1}{2}}\,\overline{|1}\,\overline{|\frac{3}{2}}}{\overline{|1}\,\overline{|\frac{3}{2}}\,\overline{|2}}\mathrm{A}=-\frac{\sqrt{\pi}}{a^3}\mathrm{A},\ \mathrm{A}_4=\frac{1}{a^4}\frac{\overline{|\frac{1}{2}}\,\overline{|1}\,\overline{|\frac{3}{2}}\,\overline{|2}}{\overline{|1}\,\overline{|\frac{3}{2}}\,\overline{|2}\,\overline{|\frac{5}{2}}}=\frac{1}{a^4}\,\frac{2^2}{1\,.\,3}\mathrm{A}$$

$$\mathrm{A}_5=-\frac{\sqrt{\pi}}{a^5}\cdot\frac{\overline{|2}\,\overline{|\frac{5}{2}}}{\overline{|\frac{5}{2}}\,\overline{|3}}\mathrm{A}=-\frac{\sqrt{\pi}}{a^5}\,\frac{1}{1\,.\,2}\mathrm{A},\ \mathrm{A}_6=\frac{1}{a^6}\,\frac{2^2}{1\,.\,3}\mathrm{A}\frac{\overline{|\frac{5}{2}}\,\overline{|3}}{\overline{|3}\,\overline{|\frac{7}{2}}}=\frac{1}{a^6}\,\frac{2^3}{1\,.\,3\,.\,5}\mathrm{A}$$

&c., &c., so that

$$y=\mathrm{A}\left\{1+\frac{2}{a^2x}+\frac{2^2}{1\,.\,3\,a^4x^2}+\frac{2^3}{1\,.\,3\,.\,5\,a^6x^3}+\text{\&c.}\right.$$

$$\left.-\sqrt{\pi}\left(\frac{1}{a\sqrt{x}}+\frac{1}{1\,.\,a^3x^{\frac{3}{2}}}+\frac{1}{1\,.\,2\,a^5x^{\frac{5}{2}}}+\text{\&c.}\right)\right\}$$

Let $$y_1=1+\frac{2}{a^2x}+\frac{2^2}{1\,.\,3\,a^4x^2}+\text{\&c.}$$

then $$\frac{d}{dx}(\sqrt{x}\cdot y_1)=\frac{1}{2\sqrt{x}}-\frac{1}{a^2x^{\frac{3}{2}}}-\text{\&c.}$$

$$=\frac{1}{2\sqrt{x}}-\frac{1}{a^2x^{\frac{3}{2}}}y_1$$

or $$\frac{dy_1}{dx}+\left(\frac{1}{2x}+\frac{1}{a^2x^2}\right)y_1=\frac{1}{2x}.$$

Again, let $$y_2 = \frac{1}{\sqrt{x}} + \frac{1}{1 \,.\, a^3 x^{\frac{3}{2}}} + \&c.$$

then $$\frac{d}{dx}(\sqrt{x}\, y_2) = -\frac{1}{1 \,.\, a^3 x^2} - \&c.$$

$$= -\frac{1}{a^2 x^{\frac{3}{2}}} y_2$$

or $$\frac{dy_2}{dx} + \left(\frac{1}{2x} + \frac{1}{a^2 x^2}\right) y_2 = 0$$

and $$y = \mathrm{A}\left(y_1 - \frac{\sqrt{\pi}}{a} y_2\right).$$

By solving the equations for y_1 and y_2 we obtain finally

$$y = \frac{\mathrm{A}\, e^{\frac{1}{a^2 x}}}{\sqrt{x}} \left\{ \frac{1}{2} \int \frac{e^{-\frac{1}{a^2 x}}}{\sqrt{x}} \, dx - \frac{\sqrt{\pi}}{a} \right\}$$

The equations from which y_1 and y_2 are determined differ only in the term which does not contain y; and it will be seen hereafter that similar equations serve to give the solution of the other differential equations of this class, when n is an integer. If $a\sqrt{-1} = m$, these equations are

$$\frac{dy_1}{dx} + \left(\frac{1}{2x} - \frac{1}{m^2 x^2}\right) y_1 = \frac{1}{2x}$$

$$\frac{dy_2}{dx} + \left(\frac{1}{2x} - \frac{1}{m^2 x^2}\right) y_2 = 0.$$

12. OTHERWISE. The following method of solving this equation has the advantage of not appearing to take for granted the form in which y is expressed in terms of x.

$$y - a\sqrt{-1}\, x \frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = 0 \text{ gives } y = \frac{0}{1 - a\sqrt{-1}\, x\, d^{\frac{1}{2}}}$$

$$= \frac{1 + a\sqrt{-1}\, x\, d^{\frac{1}{2}}}{1 + a^2 x\, d^{\frac{1}{2}} x\, d^{\frac{1}{2}}} \,.\, 0$$

Now $\dfrac{0}{1 + a^2 x\, d^{\frac{1}{2}} x\, d^{\frac{1}{2}}}$ is the solution of the equation

$$v + a^2 x\, d^{\frac{1}{2}} x\, d^{\frac{1}{2}} v = 0, \text{ or of}$$

$$\frac{v}{a^2} + x^2 \frac{dv}{dx} + \frac{1}{2} x v = 0,$$

or of $$\frac{dv}{dx} + \left(\frac{1}{2x} + \frac{1}{a^2 x^2}\right) v = 0:$$

which is the equation for determining y_2 given above.

$$\therefore \qquad v=\frac{\mathrm{A}}{\sqrt{x}}e^{\frac{1}{a^2 x}}:$$

and
$$y=(1+a\sqrt{-1}\,x\,d^{\frac{1}{2}})\,v$$

$$=\frac{\mathrm{A}}{\sqrt{x}}e^{\frac{1}{a^2 x}}+a\sqrt{-1}\,x\frac{d^{\frac{1}{2}}v}{dx^{\frac{1}{2}}}$$

which will be seen to coincide with the solution already given.

This second method of solving the equation is by far the most simple and satisfactory, when once the principles of the calculus of operations are thoroughly mastered. For the purpose, however, of exhibiting the analogy amongst the differential equations which determine the values of the different series which make up a function satisfying the conditions $y-mx^2\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0$, I shall employ the first method in the three following examples.

13. Ex. 4.
$$y-mx^{\frac{3}{2}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0.$$

let
$$y=\mathrm{A}_0+\frac{\mathrm{A}_1}{x}+\frac{\mathrm{A}_2}{x^2}+\&c.$$

then
$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=(-1)^{\frac{1}{2}}\left\{\frac{\left|\overline{\tfrac{3}{2}}\right.}{\left|\overline{1}\right.}\,\frac{\mathrm{A}_1}{x^{\frac{3}{2}}}+\frac{\left|\overline{2}\right.}{\left|\overline{\tfrac{3}{2}}\right.}\,\frac{\mathrm{A}_2}{x^{\frac{5}{2}}}-\&c.\right.$$

and
$$\mathrm{A}_0+\frac{\mathrm{A}_1}{x}+\frac{\mathrm{A}_2}{x^2}+\&c. \qquad =m(-1)^{\frac{1}{2}}\left\{\frac{\left|\overline{\tfrac{3}{2}}\right.}{\left|\overline{1}\right.}\mathrm{A}_1+\frac{\left|\overline{\tfrac{5}{2}}\right.}{\left|\overline{2}\right.}\,\frac{\mathrm{A}_2}{x}+\&c.\right\}$$

$$\mathrm{A}_1=\frac{\left|\overline{1}\right.}{\left|\overline{\tfrac{3}{2}}\right.}\,\frac{\mathrm{A}_0}{m\sqrt{-1}}\,; \quad \mathrm{A}_2=\frac{\left|\overline{2}\right.}{\left|\overline{\tfrac{5}{2}}\right.}\,\frac{\mathrm{A}_1}{m\sqrt{-1}}\,; \quad \mathrm{A}_3=\frac{\left|\overline{3}\right.}{\left|\overline{\tfrac{7}{2}}\right.}\,\frac{\mathrm{A}_2}{m\sqrt{-1}}\ \&c.$$

$$\text{or}\quad \mathrm{A}_1=\frac{2}{1}\cdot\frac{\mathrm{A}_0}{m\sqrt{-\pi}}\,; \quad \mathrm{A}_2=-\frac{2^2}{3\,.\,1}\cdot\frac{2}{1}\,\frac{1\,.\,\mathrm{A}_0}{m^2\pi}$$

$$\mathrm{A}_3=-\frac{2^3}{5\,.\,3\,.\,1}\cdot\frac{2^2}{3\,.\,1}\cdot\frac{2}{1}\cdot\frac{1\,.\,2\,\mathrm{A}_0}{m^3\pi\sqrt{-1}}\,; \quad \mathrm{A}_4=\frac{2^4}{7\,.\,5\,.\,3\,.\,1}\cdot\frac{2^3}{5\,.\,3\,.\,1}\cdot\frac{2^2}{3\,.\,1}\cdot\frac{2}{1}\,1\,.\,2\,\frac{1\,.\,2\,.\,3\,.\,\mathrm{A}_0}{m^4\pi^2}$$

&c. = &c.

$$\text{and}\quad y=\mathrm{A}_0\left\{1+\frac{2}{1}\,\frac{1}{mx\sqrt{-\pi}}-\frac{2^2}{3\,.\,1}\cdot\frac{2}{1}\cdot\frac{1}{m^2x^2\pi}-\frac{2^3}{5\,.\,3\,.\,1}\cdot\frac{2^2}{3\,.\,1}\cdot\frac{2}{1}\,\frac{1\,.\,2}{m^3x^3\sqrt{-\pi}}\ \&c.\right\}$$

Ex. 5.
$$y-mx^2\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0.$$

It is easily seen that the form of the series into which y may be expanded is this

$$y=\mathrm{A}x+\frac{\mathrm{B}}{\sqrt{x}}+\frac{\mathrm{C}}{x^2}+\frac{\mathrm{D}}{x^{\frac{7}{2}}}+\ \&c.$$

$$+\alpha+\frac{\beta}{x^{\frac{3}{2}}}+\frac{\gamma}{x^3}+\frac{\delta}{x^{\frac{9}{2}}}+\ \&c.$$

and that the result of substitution is

$$\left.\begin{array}{l} \mathrm{A}\,x+\dfrac{\mathrm{B}}{\sqrt{x}}+\dfrac{\mathrm{C}}{x^{2}}+\dfrac{\mathrm{D}}{x^{\frac{7}{2}}}+\ \&c. \\ +\,a+\dfrac{\beta}{x^{\frac{3}{2}}}+\dfrac{\gamma}{x^{3}}+\dfrac{\delta}{x^{\frac{9}{2}}}+\ \&c. \end{array}\right\}=\begin{array}{l} m\sqrt{-1}\left(\dfrac{\overline{|1}}{\overline{|\frac{1}{2}}}\,\mathrm{B}\,x+\dfrac{\overline{|\frac{5}{2}}}{\overline{|2}}\,\dfrac{\mathrm{C}}{\sqrt{x}}+\dfrac{\overline{|4}}{\overline{|\frac{7}{2}}}\,\dfrac{\mathrm{D}}{x^{2}}+\ \&c.\right) \\ +\,m\sqrt{-1}\left(\dfrac{\overline{|2}}{\overline{|\frac{3}{2}}}\,\beta+\dfrac{\overline{|\frac{7}{2}}}{\overline{|3}}\,\dfrac{\gamma}{x^{\frac{3}{2}}}+\dfrac{\overline{|5}}{\overline{|\frac{9}{2}}}\,\dfrac{\delta}{x^{3}}+\ \&c.\right) \end{array}$$

so that

$$\mathrm{B}=\frac{\overline{|\frac{1}{2}}}{\overline{|1}}\,\frac{\mathrm{A}}{m\sqrt{-1}};\quad \mathrm{C}=\frac{\overline{|2}}{\overline{|\frac{5}{2}}}\,\frac{\mathrm{B}}{m\sqrt{-1}}=-\frac{\overline{|\frac{1}{2}}}{\overline{|1}}\,\frac{\overline{|2}}{\overline{|\frac{5}{2}}}\,\frac{1}{m^{2}}$$

$$\mathrm{D}=\frac{\overline{|\frac{7}{2}}}{\overline{|4}}\,\frac{\mathrm{C}}{m\sqrt{-1}}=-\frac{\overline{|\frac{1}{2}}}{\overline{|1}}\,\frac{\overline{|2}}{\overline{|\frac{5}{2}}}\,\frac{\overline{|\frac{7}{2}}}{\overline{|4}}\,\frac{1}{m^{3}\sqrt{-1}}$$

$$\mathrm{E}=\frac{\overline{|\frac{1}{2}}}{\overline{|1}}\,\frac{\overline{|2}}{\overline{|\frac{5}{2}}}\,\frac{\overline{|\frac{7}{2}}}{\overline{|4}}\,\frac{\overline{|5}}{\overline{|\frac{11}{2}}}\,\frac{1}{m^{4}}\ \&c.$$

and also

$$\beta=\frac{\overline{|\frac{3}{2}}}{\overline{|2}}\,\frac{a}{m\sqrt{-1}},\quad \gamma=-\frac{\overline{|\frac{3}{2}}}{\overline{|2}}\,\frac{\overline{|3}}{\overline{|\frac{7}{2}}}\,\frac{a}{m^{2}},\quad \delta=-\frac{\overline{|\frac{3}{2}}}{\overline{|2}}\,\frac{\overline{|3}}{\overline{|\frac{7}{2}}}\,\frac{\overline{|\frac{9}{2}}}{\overline{|5}}\,\frac{a}{m^{3}\sqrt{-1}}$$

$$\epsilon=\frac{\overline{|\frac{3}{2}}}{\overline{|2}}\,\frac{\overline{|3}}{\overline{|\frac{7}{2}}}\,\frac{\overline{|\frac{9}{2}}}{\overline{|5}}\,\frac{\overline{|6}}{\overline{|\frac{13}{2}}}\,\frac{a}{m^{4}}\ \&c.$$

$$\therefore\quad y=\mathrm{A}\left\{x-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\,m^{2}x^{2}}+\frac{1\cdot4}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\,m^{4}x^{5}}-\frac{1\cdot4\cdot7}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\cdot\frac{13}{2}\cdot\frac{15}{2}\,m^{6}x^{8}}+\ \&c.\right.$$

$$\left.+\frac{\sqrt{\pi}}{m\sqrt{-1}}\left(\frac{1}{\sqrt{x}}-\frac{\frac{5}{2}}{2\cdot3\,m^{2}x^{\frac{7}{2}}}+\frac{\frac{5}{2}\cdot\frac{11}{2}}{2\cdot3\cdot5\cdot6\,m^{4}x^{6\frac{1}{2}}}-\frac{\frac{5}{2}\cdot\frac{11}{2}\cdot\frac{17}{2}}{2\cdot3\cdot5\cdot6\cdot8\cdot9\,m^{6}x^{9\frac{1}{2}}}+\ \&c.\right)\right\}$$

$$+\,a\left\{1-\frac{2}{\frac{3}{2}\cdot\frac{5}{2}\,m^{4}x^{3}}+\frac{2\cdot5}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\,m^{4}x^{6}}-\frac{2\cdot5\cdot8}{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{9}{2}\cdot\frac{11}{2}\cdot\frac{15}{2}\cdot\frac{17}{2}\,m^{6}x^{9}}+\ \&c.\right.$$

$$\left.+\frac{\sqrt{\pi}}{m\sqrt{-1}}\left(\frac{\frac{1}{2}}{x^{\frac{3}{2}}}-\frac{\frac{1}{2}\cdot\frac{7}{2}}{3\cdot4\,m^{2}x^{\frac{9}{2}}}+\frac{\frac{1}{2}\cdot\frac{7}{2}\cdot\frac{13}{2}}{3\cdot4\cdot6\cdot7\,m^{4}x^{\frac{15}{2}}}-\ \&c.\right)\right\}$$

Each of these four series is the integral of a differential equation of the second order.

Let
$$\frac{dy_1}{dx}=x-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\,m^{2}x^{2}}+\frac{1\cdot4}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}\,m^{4}x^{5}}-\ \&c.$$

then
$$y_1=\frac{x^{2}}{2}+\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\,m^{2}x}-\ \&c.$$

and
$$\frac{d^{2}\sqrt{x}\,y_1}{dx^{2}}=\frac{5}{2}\cdot\frac{3}{2}\,\frac{x^{\frac{1}{2}}}{2}+\frac{1}{m^{2}x^{\frac{5}{2}}}-\frac{1}{\frac{1}{2}\cdot\frac{3}{2}\,m^{4}x^{\frac{11}{2}}}+\ \&c.$$

$$=\frac{15}{8}\sqrt{x}+\frac{1}{m^{2}x^{\frac{1}{2}}}\,\frac{dy_1}{dx}$$

or
$$\frac{d^{2}y_1}{dx^{2}}+\left(\frac{1}{x}-\frac{1}{m^{2}x^{4}}\right)\frac{dy_1}{dx}-\frac{y_1}{4x^{2}}=\frac{15}{8}$$

Again, let
$$\frac{dy_2}{dx}=\frac{1}{\sqrt{x}}-\frac{\frac{5}{2}}{2\,.\,3\,m^2 x^{\frac{7}{2}}}+\frac{\frac{5}{2}\,.\,\frac{13}{2}}{2\,.\,3\,.\,5\,.\,6\,m^4 x^{\frac{13}{2}}}-\text{\&c.}$$

then
$$y_2=2\sqrt{x}+\frac{1}{2\,.\,3\,m^2 x^{\frac{5}{2}}}-\frac{\frac{5}{2}}{2\,.\,3\,.\,5\,.\,6\,m^4 x^{\frac{11}{2}}}+\text{\&c.}$$

and
$$\frac{d^2\sqrt{x}\,y_2}{dx^2}=\frac{1}{m^2 x^4}-\frac{\frac{5}{2}}{2\,.\,3\,m^4 x^7}+\text{\&c.}$$
$$=\frac{1}{m^2 x^{\frac{7}{2}}}\,\frac{dy_2}{dx}$$
$$\therefore\qquad \frac{d^2 y_2}{dx^2}+\left(\frac{1}{x}-\frac{1}{m^2 x^4}\right)\frac{dy_2}{dx}-\frac{y_2}{4x^2}=0$$

Also let
$$\frac{dy_3}{dx}=1-\frac{2}{\frac{3}{2}\,.\,\frac{5}{2}\,m^2 x^3}+\frac{2\,.\,5}{\frac{3}{2}\,.\,\frac{5}{2}\,.\,\frac{9}{2}\,.\,\frac{13}{2}\,m^4 x^6}-\text{\&c.}$$

then
$$y_3=x+\frac{1}{\frac{3}{2}\,.\,\frac{5}{2}\,m^2 x^2}-\frac{2}{\frac{3}{2}\,.\,\frac{5}{2}\,.\,\frac{9}{2}\,.\,\frac{13}{2}\,m^4 x^5}+\text{\&c.}$$

and
$$\frac{d^2\sqrt{x}\,y_3}{dx^2}=\frac{3}{2}\,\frac{1}{2\sqrt{x}}+\frac{1}{m^2 x^{\frac{7}{2}}}-\frac{2}{\frac{3}{2}\,.\,\frac{5}{2}\,m^4 x^{\frac{13}{2}}}+\text{\&c.}$$
$$=\frac{3}{4\sqrt{x}}+\frac{1}{m^2 x^{\frac{7}{2}}}\,\frac{dy_3}{dx}$$

or
$$\frac{d^2 y_3}{dx^2}+\left(\frac{1}{x}-\frac{1}{m^2 x^4}\right)\frac{dy_3}{dx}-\frac{y_3}{4x^2}=\frac{3}{4x}$$

Lastly, let
$$\frac{dy_4}{dx}=\frac{\frac{1}{2}}{x^{\frac{3}{2}}}-\frac{\frac{1}{2}\,.\,\frac{7}{2}}{3\,.\,4\,m^2 x^{\frac{9}{2}}}+\frac{\frac{1}{2}\,.\,\frac{7}{2}\,.\,\frac{13}{2}}{3\,.\,4\,.\,6\,.\,7\,m^4 x^{\frac{15}{2}}}+\text{\&c.}$$

then
$$y_4=-\frac{1}{x^{\frac{1}{2}}}+\frac{\frac{1}{2}}{3\,.\,4\,m^2 x^{\frac{7}{2}}}-\frac{\frac{1}{2}\,.\,\frac{7}{2}}{3\,.\,4\,.\,6\,.\,7\,m^4 x^{\frac{13}{2}}}+\text{\&c.}$$

and
$$\frac{d^2\sqrt{x}\,y_4}{dx^2}=\frac{\frac{1}{2}}{m^2 x^5}-\frac{\frac{1}{2}\,.\,\frac{7}{2}}{3\,.\,4\,m^4 x^8}+\text{\&c.}$$
$$=\frac{1}{m^2 x^{\frac{7}{2}}}\,\frac{dy_4}{dx}$$

or
$$\frac{d^2 y_4}{dx^2}+\left(\frac{1}{x}-\frac{1}{m^2 x^4}\right)\frac{dy_4}{dx}-\frac{y_4}{4x^2}=0$$

Having found y_1, y_2, y_3, y_4 from these equations, we obtain
$$y=\mathrm{A}\left(\frac{dy_1}{dx}+\frac{\sqrt{\pi}}{m\sqrt{-1}}\,\frac{dy_2}{dx}\right)+a\left(\frac{dy_3}{dx}+\frac{\sqrt{\pi}}{m\sqrt{-1}}\,\frac{dy_4}{dx}\right)$$

The remarkable similarity between the equations which determine y_1, y_2, y_3, y_4 leads us to conclude that the form of this function is common to all similar equations. It may be seen that the equations for y_2 and y_4 are identical: the arbitrary constants must, however, be determined differently in the two: the one function vanishes when $x=\infty$, the other does not. By solving the equations in a more general form, and by a more purely symbolical method, we shall be able to

see the reason of this analogy. We shall, in Example 7, exhibit a complete and general solution of all equations of this form.

Ex. 6.
$$y - m\,x^3 \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = 0$$

Let
$$y = \mathrm{A}\,x^2 + \mathrm{B}\,x + \mathrm{C} + \frac{\mathrm{A}_1}{x^{\frac{1}{2}}} + \frac{\mathrm{B}_1}{x^{\frac{3}{2}}} + \frac{\mathrm{C}_1}{x^{\frac{5}{2}}} + \frac{\mathrm{A}_2}{x^3} + \frac{\mathrm{B}_2}{x^4}$$
$$+ \frac{\mathrm{C}_2}{x^5} + \frac{\mathrm{A}_3}{x^{\frac{11}{2}}} + \&c.$$

then
$$\mathrm{A}\,x^2 + \mathrm{B}\,x + \mathrm{C} + \frac{\mathrm{A}_1}{x^{\frac{1}{2}}} + \&c. = m\sqrt{-1}\left(\frac{\left|\overline{1}\right.}{\left|\overline{\tfrac{1}{2}}\right.}\, x^2\,\mathrm{A}_1 + \&c.\right)$$

$$\mathrm{A}_1 = \frac{\mathrm{A}}{m\sqrt{-1}} \frac{\left|\overline{\tfrac{1}{2}}\right.}{\left|\overline{1}\right.},\ \mathrm{A}_2 = \frac{\mathrm{A}_1}{m\sqrt{-1}} \frac{\left|\overline{3}\right.}{\left|\overline{3\tfrac{1}{2}}\right.},\ \mathrm{A}_3 = \frac{\mathrm{A}_2}{m\sqrt{-1}} \frac{\left|\overline{5\tfrac{1}{2}}\right.}{\left|\overline{6}\right.}\ \&c.$$

$$\mathrm{B}_1 = \frac{\mathrm{B}}{m\sqrt{-1}} \frac{\left|\overline{\tfrac{3}{2}}\right.}{\left|\overline{2}\right.},\ \mathrm{B}_2 = \frac{\mathrm{B}_1}{m\sqrt{-1}} \frac{\left|\overline{4}\right.}{\left|\overline{4\tfrac{1}{2}}\right.},\ \mathrm{B}_3 = \frac{\mathrm{B}_2}{m\sqrt{-1}} \frac{\left|\overline{6\tfrac{1}{2}}\right.}{\left|\overline{7}\right.}\ \&c.$$

$$\mathrm{C}_1 = \frac{\mathrm{C}}{m\sqrt{-1}} \frac{\left|\overline{\tfrac{5}{2}}\right.}{\left|\overline{3}\right.},\ \mathrm{C}_2 = \frac{\mathrm{C}_1}{m\sqrt{-1}} \frac{\left|\overline{5}\right.}{\left|\overline{5\tfrac{1}{2}}\right.},\ \mathrm{C}_3 = \frac{\mathrm{C}_2}{m\sqrt{-1}} \frac{\left|\overline{7\tfrac{1}{2}}\right.}{\left|\overline{8}\right.}\ \&c.$$

This gives us six separate series.

1°.
$$\mathrm{A}\,x^2 + \frac{\mathrm{A}_2}{x^3} + \frac{\mathrm{A}_4}{x^8} + \&c. = \mathrm{A}\left(x^2 - \frac{1\,.\,2}{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}\,m^2 x^3} + \frac{1\,.\,2\,.\,6\,.\,7}{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}\,.\,\frac{11}{2}\,.\,\frac{13}{2}\,.\,\frac{15}{2}\,m^4 x^8} + \&c.\right)$$

Let
$$\frac{d^2 y_1}{d x^2} = x^2 - \frac{1\,.\,2}{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}\,m^2 x^3} + \&c.$$

then
$$y_1 = \frac{x^4}{3\,.\,4} - \frac{1}{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}\,m^2 x} + \frac{1\,.\,2}{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}\,.\,\frac{11}{2}\,.\,\frac{13}{2}\,.\,\frac{15}{2}\,m^4 x^6} - \&c.$$

$$\frac{d^3 \sqrt{x}\,.\,y_1}{d x^3} = \frac{\frac{9}{2}\,.\,\frac{7}{2}\,.\,\frac{5}{2}\,x^{\frac{3}{2}}}{3\,.\,4} + \frac{1}{m^2 x^{\frac{1}{2}}} - \frac{1\,.\,2}{\frac{1}{2}\,.\,\frac{3}{2}\,.\,\frac{5}{2}\,m^4 x^{\frac{11}{2}}} + \&c.$$

$$= \frac{105}{32}\,x^{\frac{3}{2}} + \frac{1}{m^2 x^{\frac{11}{2}}} \frac{d^2 y_1}{d x^2}$$

or
$$\frac{d^3 y_1}{d x^3} + \frac{3}{2\,x}\frac{d^2 y_1}{d x^2} - \frac{3}{4\,x^2}\frac{d y_1}{d x} + \frac{3}{8\,x^3}\,y_1 = \frac{105}{32}\,x + \frac{1}{m^2 x^6}\frac{d^2 y_1}{d x^2}$$

or
$$\frac{d^3 y_1}{d x^3} + \left(\frac{3}{2\,x} - \frac{1}{m^2 x^6}\right)\frac{d^2 y_1}{d x^2} - \frac{3}{4\,x^2}\frac{d y_1}{d x} + \frac{3}{8\,x^3}\,y_1 = \frac{105}{32}\,x$$

2°. $\frac{\mathrm{A}_1}{x^{\frac{1}{2}}} + \frac{\mathrm{A}_3}{x^{\frac{11}{2}}} + \&c.$, gives
$$\frac{\sqrt{\pi}}{m\sqrt{-1}}\,\mathrm{A}\left(\frac{1}{x^{\frac{1}{2}}} - \frac{\frac{7}{2}\,.\,\frac{9}{2}}{3\,.\,4\,.\,5\,m^2 x^{\frac{11}{2}}} + \frac{\frac{7}{2}\,.\,\frac{9}{2}\,.\,\frac{17}{2}\,.\,\frac{19}{2}}{3\,.\,4\,.\,5\,.\,8\,.\,9\,.\,10\,m^2 x^{\frac{21}{2}}} + \&c.\right)$$

Let
$$\frac{d^2 y_2}{d x^2} = \frac{1}{x^{\frac{1}{2}}} - \frac{\frac{7}{2}\,.\,\frac{9}{2}}{3\,.\,4\,.\,5\,m^2 x^{\frac{11}{2}}} + \&c.$$

then $y_2 = 2 \cdot \frac{2}{3} x^{\frac{3}{2}} - \dfrac{1}{3 \,.\, 4 \,.\, 5\, m^2 x^{\frac{7}{2}}} + \dfrac{\frac{7}{2} \cdot \frac{9}{2}}{3 \,.\, 4 \,.\, 5 \,.\, 8 \,.\, 9 \,.\, 10\, m^4 x^{\frac{19}{2}}} - \&c.$

$$\frac{d^3\, y_2 \sqrt{x}}{d\,x^3} = \frac{1}{m^2 x^6} - \frac{\frac{7}{2} \cdot \frac{9}{2}}{3 \,.\, 4 \,.\, 5\, m^4 x^{11}} + \&c.$$

$$= \frac{1}{m^2 x^{\frac{11}{2}}} \frac{d^2 y_2}{d\,x^2}$$

$$\therefore \quad \frac{d^3 y_2}{d\,x^3} + \left(\frac{3}{2\,x} - \frac{1}{m^2 x^6}\right) \frac{d^2 y_2}{d\,x^2} - \frac{3}{4\,x^2} \frac{d\,y_2}{d\,x} + \frac{3}{8\,x^3} y_2 = 0$$

3°. $\mathrm{B}\,x + \dfrac{\mathrm{B}_2}{x^4} + \dfrac{\mathrm{B}_3}{x^9} + \&c. = \mathrm{B}\left(x - \dfrac{2 \,.\, 3}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}\, m^2 x^4} + \dfrac{2 \,.\, 3 \,.\, 7 \,.\, 8}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{13}{2}\, \frac{15}{2} \cdot \frac{17}{2}\, m^4 x^9} + \&c.\right)$

$$= \mathrm{B} \frac{d^2 y_3}{d\,x^2} \text{ suppose}$$

then $y_3 = \dfrac{x^3}{2 \,.\, 3} - \dfrac{1}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}\, m^2 x^2} + \dfrac{2 \,.\, 3}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2} \cdot \frac{13}{2} \cdot \frac{15}{2} \cdot \frac{17}{2}\, m^4 x^7} - \&c.$

$$\frac{d^3 \sqrt{x}\, y_3}{d\,x^3} = \frac{\frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}\, x^{\frac{1}{2}}}{2 \,.\, 3} + \frac{1}{m^2 x^{\frac{9}{2}}} - \frac{2 \,.\, 3}{\frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}\, m^4 x^{\frac{19}{2}}} + \&c.$$

$$= \frac{35}{16} x^{\frac{1}{2}} + \frac{1}{m^2 x^{\frac{11}{2}}} \frac{d^2 y_3}{d\,x^2}$$

$$\therefore \quad \frac{d^3 y_3}{d\,x^3} + \left(\frac{3}{2\,x} - \frac{1}{m^2 x^6}\right) \frac{d^2 y_3}{d\,x^2} - \frac{3}{4\,x^2} \frac{d\,y_3}{d\,x} + \frac{3}{8\,x^3} y_3 = \frac{35}{16}$$

4°. $\dfrac{\mathrm{B}_1}{x^{\frac{3}{2}}} + \dfrac{\mathrm{B}_3}{x^{\frac{13}{2}}} + \dfrac{\mathrm{B}_5}{x^{\frac{23}{2}}} + \&c. = \dfrac{\mathrm{B}\sqrt{\pi}}{m\sqrt{-1}} \left(\dfrac{\frac{1}{2}}{x^{\frac{3}{2}}} - \dfrac{\frac{1}{2} \cdot \frac{9}{2} \cdot \frac{11}{2}}{4 \,.\, 5 \,.\, 6\, m^2 x^{\frac{13}{2}}} + \&c.\right)$

$$= \frac{\sqrt{\pi}}{m\sqrt{-1}} \mathrm{B}_1 \frac{d^2 y_4}{d\,x^2} \text{ suppose}$$

then $y_4 = -2\, x^{\frac{1}{2}} - \dfrac{\frac{1}{2}}{4 \,.\, 5 \,.\, 6\, m^2 x^{\frac{9}{2}}} + \&c.$

$\dfrac{d^3 \sqrt{x}\, y_4}{d\,x^3} = \dfrac{1}{m^2 x^7} - \&c. = \dfrac{1}{m^2 x^{\frac{11}{2}}} y_4$ the same equation as for y_2.

5°. $\mathrm{C} + \dfrac{\mathrm{C}_2}{x^5} + \dfrac{\mathrm{C}_4}{x^{10}} + \&c. = \mathrm{C}\left(1 - \dfrac{3 \,.\, 4}{\frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2}\, m^2 x^5} + \&c.\right)$

$$= \mathrm{C} \frac{d^2 y_5}{d\,x^2}$$

$$\therefore \quad y_5 = \frac{x^2}{1 \,.\, 2} - \frac{1}{\frac{5}{2} \cdot \frac{7}{2} \cdot \frac{9}{2}\, m^2 x^3} + \&c.$$

$$\frac{d^3 \sqrt{x}\, y_5}{d\,x^3} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{m^2 x^{\frac{11}{2}}} - \&c.$$

$$= \frac{15}{8\sqrt{x}} + \frac{1}{m^2 x^{\frac{11}{2}}} \frac{d^2 y_5}{d\,x^2}$$

$$\frac{d^3 y_5}{d\,x^3} + \left(\frac{3}{2\,x} - \frac{1}{m^2 x^6}\right) \frac{d^2 y_5}{d\,x^2} - \frac{3}{4\,x^2} \frac{d\,y_5}{d\,x} + \frac{3}{8\,x^3} y_5 = \frac{15}{8\,x}$$

6°. $$\frac{C_1}{x^{\frac{5}{2}}} + \frac{C_3}{x^{\frac{15}{2}}} + \&c. = \frac{\sqrt{\pi}\,C}{m\sqrt{-1}}\left(\frac{\frac{1}{2}\cdot\frac{3}{2}}{2\,x^{\frac{5}{2}}} - \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{11}{2}\cdot\frac{13}{2}}{2\,.\,5\,.\,6\,.\,7}\,\frac{1}{m^2\,x^{\frac{15}{2}}} + \&c.\right)$$

$$= \frac{\sqrt{\pi}\,C}{m\sqrt{-1}}\,\frac{d^2 y_6}{d\,x^2}$$

$\therefore$ $$y_6 = \frac{1}{2\sqrt{x}} - \frac{\frac{1}{2}\cdot\frac{3}{2}}{2\,.\,5\,.\,6\,.\,7}\,\frac{1}{m^2\,x^{\frac{11}{2}}} + \&c.$$

$$\frac{d^3\sqrt{x}\,y_6}{d\,x^3} = \frac{\frac{1}{2}\cdot\frac{3}{2}}{2\,m^2\,x^8} - \&c.$$

$$= \frac{1}{m^2\,x^{\frac{11}{2}}}\,\frac{d^2 y_6}{d\,x^2} \text{ the same as } y_2\,;$$

and $$y = A\left(\frac{d^2 y_1}{d\,x^2} + \frac{\sqrt{\pi}}{m\sqrt{-1}}\,\frac{d^2 y_2}{d\,x^2}\right) + B\left(\frac{d^2 y_3}{d\,x^2} + \frac{\sqrt{\pi}}{m\sqrt{-1}}\,\frac{d^2 y_4}{d\,x^2}\right)$$

$$+ C\left(\frac{d^2 y_5}{d\,x^2} + \frac{\sqrt{\pi}}{m\sqrt{-1}}\,\frac{d^2 y_6}{d\,x^2}\right)$$

It is scarcely necessary to point out the analogy which exists between the differential equations which determine the value of the transcendentals in this and in the preceding examples.

14. We proceed now to exhibit a general solution of equations of this kind.

Ex. 7. $$y - m\,x^n\,\frac{d^{\frac{1}{2}} y}{d\,x^{\frac{1}{2}}} = 0\,;\ n \textit{ being any integer.}$$

The symbolical form of this equation is

$$y = \frac{1}{1 - m\,x^n\,d^{\frac{1}{2}}}\,.\,0 = \frac{1 + m\,x^n\,d^{\frac{1}{2}}}{1 - m^2\,x^n\,d^{\frac{1}{2}}\,x^n\,d^{\frac{1}{2}}}\,0$$

$$= (1 + m\,x^n\,d^{\frac{1}{2}})\,v = v + m\,x^n\,\frac{d^{\frac{1}{2}} v}{d\,x^{\frac{1}{2}}} \qquad (1)$$

where v is determined by the equation

$$\frac{1}{1 - m^2\,x^n\,d^{\frac{1}{2}}\,x^n\,d^{\frac{1}{2}}}\,0 = v\,; \text{ or}$$

$$v - m^2\,x^n\,d^{\frac{1}{2}}\,x^n\,d^{\frac{1}{2}}\,v = 0 \text{ or } v - m^2\,x^n\,\frac{d^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}}\left(x^n\,\frac{d^{\frac{1}{2}} v}{d\,x^{\frac{1}{2}}}\right) = 0 \qquad (2)$$

Let $$v = \frac{d^{n-1} z}{d\,x^{n-1}}\,;\text{ then } \frac{d^{\frac{1}{2}} v}{d\,x^{\frac{1}{2}}} = \frac{d^{n-\frac{1}{2}} z}{d\,x^{n-\frac{1}{2}}}.$$

$\therefore$ $$\frac{d^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}}\left(x^n\,\frac{d^{\frac{1}{2}} v}{d\,x^{\frac{1}{2}}}\right) = x^n\,\frac{d^n z}{d\,x^n} + \frac{1}{2}\,n\,x^{n-1}\,\frac{d^{n-1} z}{d\,x^{n-1}}$$

$$- \frac{1\,.\,1}{2\,.\,4}\,n\,(n-1)\,x^{n-2}\,\frac{d^{n-2} z}{d\,x^{n-2}} + \&c. + (-1)^{n-1}\,\frac{1\,.\,1\,.\,3\ldots(2\,n-3)}{2\,.\,4\cdots 2\,n}\,n\,(n-1)\ldots 1\,.\,z$$

(Part I. Art. 11.)

By substituting this in equation (2) we obtain

$$\frac{d^{n-1} z}{d x^{n-1}} - m^2 x^n \left(x^n \frac{d^n z}{d x^n} + \frac{1}{2} n x^{n-1} \frac{d^{n-1} z}{d x^{n-1}} + \&c. \right) = 0$$

or
$$\frac{d^n z}{d x^n} + \left(\frac{n}{2x} - \frac{1}{m^2 x^{2n}} \right) \frac{d^{n-1} z}{d x^{n-1}} - \frac{1 \,.\, 1}{2 \,.\, 4} \frac{n(n-1)}{x^2} \frac{d^{n-2} z}{d x^{n-2}}$$

$$+ \frac{1 \,.\, 1 \,.\, 3}{2 \,.\, 4 \,.\, 6} \frac{n(n-1)(n-2)}{x^3} \frac{d^{n-3} z}{d x^{n-3}} - \&c. + (-1)^{n-1} \frac{1 \,.\, 1 \,.\, 3 \ldots (2n-3)}{2 \,.\, 4 \,.\, 6 \ldots 2n} \,.$$

$$. \frac{n(n-1)\ldots 1}{x^n} z = 0$$

When z has been determined from this equation, we shall have the complete value of y by means of Equation (1.), viz.

$$y = \frac{d^{n-1} z}{d x^{n-1}} + m x^n \frac{d^{n-\frac{1}{2}} z}{d x^{n-\frac{1}{2}}}$$

COR. If $n=3$;

$$\frac{d^3 z}{d x^3} + \left(\frac{3}{2x} - \frac{1}{m^2 x^6} \right) \frac{d^2 z}{d x^2} - \frac{3}{4 x^2} \frac{d z}{d x} + \frac{3}{8 x^3} z = 0;$$

which is the same equation as that which we obtained by a totally different process for determining y_2 and y_4 in Ex. 6.

Ex. 8.
$$y - m x^n \frac{d^{\frac{1}{2}} y}{d x^{\frac{1}{2}}} = X$$

The solution is

$$y = \frac{1}{1 - m x^n d^{\frac{1}{2}}} (X + 0) = \frac{1 + m x^n d^{\frac{1}{2}}}{1 - m^2 x^n d^{\frac{1}{2}} x^n d^{\frac{1}{2}}} (X + 0)$$

$$= (1 + m x^n d^{\frac{1}{2}})(v + w)$$

$$= v + m x^n \frac{d^{\frac{1}{2}} v}{d x^{\frac{1}{2}}} + w + m x^n \frac{d^{\frac{1}{2}} w}{d x^{\frac{1}{2}}}$$

where v is the same as in the last Example, and w is determined from the equation

$$w - m^2 x^n \frac{d^{\frac{1}{2}}}{d x^{\frac{1}{2}}} \left(\frac{x^n d^{\frac{1}{2}} w}{d x^{\frac{1}{2}}} \right) = X$$

or by writing $\frac{d^{n-1} u}{d x^{n-1}}$ for w, and proceeding as in the last Example,

$$\frac{d^{n-1} u}{d x^{n-1}} - m^2 x^n \left(x^n \frac{d^n u}{d x^n} + \frac{n x^{n-1}}{2} \frac{d^{n-1} u}{d x^{n-1}} + \&c. \right) = X, \text{ or}$$

$$\frac{d^n u}{d x^n}+\left(\frac{n}{2x}-\frac{1}{m^2 x^{2n}}\right)\frac{d^{n-1} u}{d x^{n-1}}-\frac{1 \,.\, 1}{2 \,.\, 4}\,\frac{n(n-1)}{x^2}\,\frac{d^{n-2} u}{d x^{n-2}}+\&c.$$

$$+(-1)^{n-1}\frac{1 \,.\, 1 \,.\, 3 \,\ldots\, (2n-3)}{2 \,.\, 4 \,.\, 6 \,\ldots\, 2n}\cdot\frac{n(n-1)\ldots 1}{x^n}\,u=-\frac{X}{m^2 x^{2n}}$$

COR. 1. If $n=1$, the equation for determining u is

$$\frac{du}{dx}+\left(\frac{1}{2x}-\frac{1}{m^2 x^2}\right)u=-\frac{X}{m^2 x^2},$$

of which the solution is
$$u=-\frac{e^{-\frac{1}{m^2 x}}}{\sqrt{x}}\int\frac{e^{\frac{1}{m^2 x}}\,X}{m^2 x^{\frac{3}{2}}}\,dx=w$$

$$\therefore \quad v=\frac{A\,e^{-\frac{1}{m^2 x}}}{\sqrt{x}}$$

and
$$y=\left(1+m\,x\,\frac{d^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}}\right)\left(\frac{A\,e^{-\frac{1}{m^2 x}}}{\sqrt{x}}-\frac{e^{-\frac{1}{m^2 x}}}{\sqrt{x}}\int\frac{e^{\frac{1}{m^2 x}}\,X\,dx}{m^2 x^{\frac{3}{2}}}\right)$$

COR. 2. If $n=1$, $X=\frac{b}{\sqrt{x}}$, it is evident that

$$u=\frac{b}{\sqrt{x}} \qquad \therefore \qquad y=y_0+\frac{b}{\sqrt{x}}+\frac{m\,b\sqrt{-1}}{\sqrt{\pi}}$$

where y_0 is the solution of the equation without X (Ex. 3.)

It appears, therefore, that the complete solution of equations of this form is reduced to the solution of ordinary linear equations, and the determination of the half differential coefficient of the results.

Ex. 9. $\quad y-m\,x^n\,\frac{d^{r+\frac{1}{2}}\,y}{d\,x^{r+\frac{1}{2}}}=X$, where n and r are any whole numbers.

We have
$$y=\frac{X+0}{1-m\,x^n\,d^{r+\frac{1}{2}}}=\frac{1+m\,x^n\,d^{r+\frac{1}{2}}}{1-m^2\,x^n\,d^{r+\frac{1}{2}}\,x^n\,d^{r+\frac{1}{2}}}\cdot(X+0)$$

$$=\left(1+m\,x^n\,\frac{d^{r+\frac{1}{2}}}{d\,x^{r+\frac{1}{2}}}\right)(v+w) \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

where $v+w$ is the solution of the equation

$$\left(1-m^2\,x^n\,\frac{d^{r+\frac{1}{2}}}{d\,x^{r+\frac{1}{2}}}\,x^n\,\frac{d^{r+\frac{1}{2}}}{d\,x^{r+\frac{1}{2}}}\right)(v+w)=X+0.$$

Now
$$\frac{d^{r+\frac{1}{2}}}{d\,x^{r+\frac{1}{2}}}\,x^n\,\frac{d^{r+\frac{1}{2}}\,v}{d\,x^{r+\frac{1}{2}}}=x^n\,\frac{d^{2r+1}\,v}{d\,x^{2r+1}}+(r+\tfrac{1}{2})\,n x^{n+1}\,\frac{d^{2r}\,v}{d\,x^{2r}}$$

$$+\frac{(r+\frac{1}{2})(r-\frac{1}{2})}{1 \,.\, 2}\,n(n-1)\,x^{n-2}\,\frac{d^{2r-1}\,v}{d\,x^{2r-1}}-\&c.$$

$$+\frac{(r+\frac{1}{2})(r-\frac{1}{2})\ldots\ldots}{1.2\ldots\ldots n}\,n(n-1)\ldots 1.\frac{d^{2r-n+1}v}{dx^{2r-n+1}}$$

$$=x^n\frac{d^n z}{dx^n}+(r+\tfrac{1}{2})\,n\,x^{n-1}\frac{d^{n-1}z}{dx^{n-1}}+\ldots.$$

$$+\frac{(r+\frac{1}{2})(r-\frac{1}{2})\ldots(r-2n+\frac{5}{2})}{1.2\ldots\ldots\ldots n}.\,n(n-1)\ldots 1.z$$

where
$$v=\frac{d^{n-(2r+1)}z}{dx^{n-(2r+1)}}$$

∴ the equation for determining v is

$$\frac{d^n z}{dx^n}+\frac{(r+\frac{1}{2})n}{x}\frac{d^{n-1}z}{dx^{n-1}}+\frac{(r+\frac{1}{2})(r-\frac{1}{2})}{1.2}.\frac{n(n-1)}{x^2}\frac{d^{n-2}z}{dx^{n-2}}+\&c.$$

$$+\frac{(r+\frac{1}{2})(r-\frac{1}{2})\ldots.(r-2n+\frac{5}{2})}{1.2\ldots\ldots\ldots n}.\,n(n-1)\ldots 1.z$$

$$-\frac{1}{m^2x^{2n}}\frac{d^{n-(2r+1)}v}{dx^{n-(2r+1)}}=-\frac{1}{m^2x^{2n}}X\quad.\quad.\quad.\quad.\quad(2)$$

w is the particular value of v corresponding with $X=0$. Having thus obtained v and w, equation (1) gives the complete value of y. It must be observed, that the transformation from v to z is only to be made when n is greater than $2r+1$.

Class III. *Equations which are capable of solution by transformation, without division of operations.*

15. Ex. 1.
$$y-mx^{\frac{3}{2}}\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}}=0$$

By (C) this equation is transformed into

$$y-m(-1)^{\frac{3}{2}}\frac{\overline{|-D+\frac{3}{2}}}{\overline{|-D}}\,y=0,\text{ or}$$

$$y=\left(1+m\sqrt{-1}\,\frac{\overline{|-D+\frac{3}{2}}}{\overline{|-D}}\right)^{-1}.\,0.$$

Hence, as in Ex. 1, Class 2, the value of y is $y=\frac{A}{x^n}$, where n is determined by the equation $1+m\sqrt{-1}\,\frac{\overline{|n+\frac{3}{2}}}{\overline{|n}}=0$.

Cor. If $m=-\frac{4}{3\sqrt{\pi}\sqrt{-1}}$, $\overline{|n+\frac{3}{2}}=\frac{3}{2}\,.\,\frac{1}{2}\sqrt{\pi}\ \overline{|n}=\overline{|\tfrac{5}{2}}\ \overline{|n}$

$$\therefore\qquad n=1;\quad\text{and}\quad y=\frac{A}{x}.$$

Ex. 2. $$y-m\,x^{\frac{3}{2}}\,\frac{d^{\frac{3}{2}}y}{d\,x^{\frac{3}{2}}}=\mathrm{X}.$$

This gives $$y=\frac{\mathrm{A}}{x^n}+\left(1+m\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\tfrac{3}{2}}}{\overline{|-\mathrm{D}}}\right)^{-1}.\ \mathrm{X}$$

$$=\frac{\mathrm{A}}{x^n}+\Sigma\,b_r\left(1+m\sqrt{-1}\,\frac{\overline{|r+\tfrac{3}{2}}}{\overline{|r}}\right)^{-1}e^{-r\theta}$$

Cor. If $r=n$, this expression must be reduced, as in Ex. 2, Class 2, to

$$y=\frac{\mathrm{A}}{x^n}+\frac{b_r}{m\sqrt{-1}}\,\frac{\log x}{x^n}\,\frac{1}{\frac{d}{d\,n}\frac{\overline{|n+\tfrac{3}{2}}}{\overline{|n}}}+\Sigma\,b_s\left(1+m\sqrt{-1}\,\frac{\overline{|s+\tfrac{3}{2}}}{\overline{|s}}\right)^{-1}.\ \frac{1}{x^s}$$

Cor. 2. As a particular case, the solution of

$$\frac{d^{\frac{3}{2}}y}{d\,x^{\frac{3}{2}}}+\frac{3}{4}\sqrt{-1}\,\sqrt{\pi}\,\frac{y}{x^{\frac{3}{2}}}=\frac{b}{x^{\frac{7}{2}}}\ \text{ is}$$

$$y=\frac{\mathrm{A}}{x}-\frac{8\,b}{9\sqrt{-1}\,\sqrt{\pi}}\cdot\frac{1}{x^2}$$

These equations might have been included in the preceding Class, to which, both in their form and in the mode of their solution, they are very analogous. They are, however, particular cases of Example 5, below, which does not belong to that Class.

Ex. 3. $$y+a\sqrt{x}\,\frac{d^{\frac{1}{2}}y}{d\,x^{\frac{1}{2}}}+b\,x\,\frac{d\,y}{d\,x}=0.$$

The equation in θ is (by C),

$$y+a\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\tfrac{1}{2}}}{\overline{|-\mathrm{D}}}\,y-b\,\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}}}\ .\ y=0$$

or $$\left\{1+a\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\tfrac{1}{2}}}{\overline{|-\mathrm{D}}}-b\,\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}}}\right\}.\ y=0$$

Suppose $y=\Sigma\,a_n\,e^{-n\theta}$; then

$$\Sigma\,a_n\left\{1+a\sqrt{-1}\,\frac{\overline{|n+\tfrac{1}{2}}}{\overline{|n}}-b\,\frac{\overline{|n+1}}{\overline{|n}}\right\}.\ e^{-n\theta}=0\ \text{ by (A)}$$

Hence any value of n which will satisfy the equation

$$1+a\sqrt{-1}\,\frac{\overline{|n+\tfrac{1}{2}}}{\overline{|n}}-b\,\frac{\overline{|n+1}}{\overline{|n}}=0$$

will give a term in the solution.

Cor. 1. If $a\sqrt{-1}=-\frac{2\sqrt{\pi}}{4-\pi}$, $b=\frac{4-2\,\pi}{4-\pi}$ we have

$$4-\pi-n\,(4-2\,\pi)-2\,\sqrt{\pi}\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}=0$$

which is satisfied by $n=\frac{1}{2}$ and $n=1$.

Hence
$$y=\frac{\mathrm{A}}{\sqrt{x}}+\frac{\mathrm{B}}{x}.$$

Cor. 2. If n be a whole number r; $\Gamma(n)=1\,.\,2\,.\,.\,.\,(r-1)$

and $\Gamma(n+\frac{1}{2})=\frac{1}{2}\,.\,\frac{3}{2}\,.\,.\,.\,(r-\frac{1}{2})\,\sqrt{\pi}$

$$\therefore\quad 1\,.\,2\,.\,.\,.\,(r-1)+a\,\sqrt{-1}\,\sqrt{\pi}\,\frac{1\,.\,3\,.\,.\,.\,(2\,r-1)}{2^r}-b\,.\,1\,.\,2\,.\,.\,.\,r=0;$$

will determine the integral values of n.

If $n=r+\frac{1}{2}$, $\Gamma(n)=\frac{1\,.\,3\,.\,.\,.\,(2\,r-1)}{2^r}\sqrt{\pi}$, $\Gamma(n+\frac{1}{2})=1\,.\,2\,.\,.\,.\,r$

and
$$\frac{1\,.\,3\,.\,.\,.\,(2\,r-1)}{2^r}\,\sqrt{\pi}+a\sqrt{-1}\,.\,1\,.\,2\,.\,.\,.\,r-b\,\frac{1\,.\,3\,.\,.\,.\,(2\,r+1)}{2^{r+1}}\,\sqrt{\pi}=0,$$

which determines the fractional values of n which have 2 as their denominator.

Now it is evident that these are the only forms which n can assume; therefore the determination of the values of n is reduced to the solution of these two equations.

Ex. 4.
$$y+a\,\sqrt{x}\,\frac{d^{\frac{1}{2}}\,y}{d\,x^{\frac{1}{2}}}+b\,x\,\frac{d\,y}{d\,x}=\mathrm{X}.$$

Let $\mathrm{X}=\Sigma\,b_r\,e^{-r\theta}$, then

$$y=\Sigma\,a_n\,e^{-n\theta}+\Sigma\,b_r\,\left(1+a\,\sqrt{-1}\,\frac{\Gamma(-\mathrm{D}+\frac{1}{2})}{\Gamma(-\mathrm{D})}-b\,\frac{\Gamma(-\mathrm{D}+1)}{\Gamma(-\mathrm{D})}\right)^{-1}.\,e^{-r\theta}$$

$$=\Sigma\,a_n\,e^{-n\theta}+\Sigma\,\frac{b_r\,x^{-r}}{1+a\,\sqrt{-1}\,\frac{\Gamma(r+\frac{1}{2})}{\Gamma(r)}-b\,\frac{\Gamma(r+1)}{\Gamma(r)}}\quad\text{by (A)}$$

the values of n being determined as in Example 3.

Cor. If $r=p$, $n=p$, we obtain, as in other instances,

$$y=\Sigma\,\frac{a_n}{x^n}+\mathrm{C}\,\frac{\log x}{x^p}+\Sigma\,\frac{b_s\,x^{-s}}{1-a\,\sqrt{-1}\,\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}-b\,\frac{\Gamma(s+1)}{\Gamma(s)}}$$

where
$$\mathrm{C}=-\,\frac{1}{a\sqrt{-1}\,\frac{d}{dp}\,\frac{\Gamma(p+\frac{1}{2})}{\Gamma(p)}-b}$$

Ex. 5.
$$x^m\,\frac{d^m\,y}{d\,x^m}+a\,x^{m-\frac{1}{2}}\,\frac{d^{m-\frac{1}{2}}\,y}{d\,x^{m-\frac{1}{2}}}+\text{\&c.}=\mathrm{X}.$$

The equation in θ is

$$\left\{(-1)^m \frac{|\overline{-D+m}}{|\overline{-D}} + a\,(-1)^{m-\frac{1}{2}} \frac{|\overline{-D+m-\frac{1}{2}}}{|\overline{-D}} + \text{\&c.}\right\} y = X$$

which may be written $f(-D)\,y = X$;

and $$y = \{f(-D)\}^{-1}\,.\,0 + \{f(-D)\}^{-1}\,.\,X$$

$$= \Sigma\, a_n x^{-n} + \Sigma \frac{b_r\, x^{-r}}{f(r)}$$

the values of n being determined by the equation $f(n)=0$.

Ex. 6. $$(a\,x+\beta)^m \frac{d^m y}{d x^m} + a\,(a\,x+\beta)^{m-\frac{1}{2}} \frac{d^{m-\frac{1}{2}} y}{d x^{m-\frac{1}{2}}} + \text{\&c.} = X.$$

Let $x' = a\,x+\beta$, then $\frac{d^m y}{d\,x^m} = a^m \frac{d^m\, y}{d\,x'^m}$ (Part 1, Art. 27.)

&c. = &c.

$$\therefore \quad a^m\, x'^m \frac{d^m y}{d\,x'^m} + a\,a^{m-\frac{1}{2}}\, x'^{m-\frac{1}{2}} \frac{d^{m-\frac{1}{2}} y}{d\,x'^{m-\frac{1}{2}}} + \text{\&c.} = X'$$

which coincides with Example 5.

Ex. 7. $$\frac{dy}{dx} - a\,.\,\frac{d^{\frac{1}{2}} y}{d\,x^{\frac{1}{2}}} - \frac{1}{2}\,\frac{y}{x} = 0$$

By multiplying by x and reducing to differentials in θ, we get

$$\frac{|\overline{-D+1}}{|\overline{-D}}\, y + a\sqrt{x}\,(-1)^{\frac{1}{2}} \frac{|\overline{-D+\frac{1}{2}}}{|\overline{-D}}\, y + \tfrac{1}{2} y = 0$$

$$\left(\frac{|\overline{-D+1}}{|\overline{-D}} + \frac{1}{2}\right) y + a\,(-1)^{\frac{1}{2}} \frac{|\overline{-D+1}}{|\overline{-D+\frac{1}{2}}}\, e^{\frac{\theta}{2}}\, y = 0$$

or $$(-D+\tfrac{1}{2})\,y + a\,(-1)^{\frac{1}{2}} \frac{|\overline{-D+1}}{|\overline{-D+\frac{1}{2}}}\, e^{\frac{\theta}{2}}\, y = 0$$

or $$y + a\,(-1)^{\frac{1}{2}} \frac{|\overline{-D+1}}{|\overline{-D+\frac{3}{2}}}\, e^{\frac{\theta}{2}}\, y = 0$$

or $$y + a\,(-1)^{\frac{1}{2}}\, e^{\frac{3\theta}{2}} \frac{|\overline{-D-\frac{1}{2}}}{|\overline{-D}}\, \frac{y}{e^{\theta}} = 0$$

or $$\frac{y}{x} - a\,(-1)^{-\frac{1}{2}}\, x^{\frac{1}{2}} \frac{|\overline{-D-\frac{1}{2}}}{|\overline{-D}}\, \frac{y}{x} = 0$$

or $$\frac{y}{x} - a\,.\,\frac{d^{-\frac{1}{2}} \frac{y}{x}}{d\,x^{-\frac{1}{2}}} = 0.$$

If $\frac{y}{x}=v$; this gives

$$v-a\frac{d^{-\frac{1}{2}}v}{dx^{-\frac{1}{2}}}=0,$$

whence $$v=\mathrm{A}\,e^{a^2 x}$$

and $$y=v\,x=\mathrm{A}\,x\,e^{a^2 x}.$$

This equation may be integrated in the following manner. The equation

$$y+a\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}+\frac{3}{2}}}\,e^{\frac{\theta}{2}}\,y=0,$$

may be made to depend on the equation

$$v+a\sqrt{-1}\,\frac{\overline{|-\mathrm{D}}}{\overline{|-\mathrm{D}+\frac{1}{2}}}\,e^{\frac{\theta}{2}}\,v=0$$

by the relation $$y=\mathrm{P}_{\frac{1}{2}}\,\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}+\frac{3}{2}}}\,\frac{\overline{|-\mathrm{D}+\frac{1}{2}}}{\overline{|-\mathrm{D}}}\,v, \text{ where}$$

$$\mathrm{P}_{\frac{1}{2}}f(\mathrm{D})=f(\mathrm{D})\,f(\mathrm{D}-\tfrac{1}{2})\,f(\mathrm{D}-1)\ \&\text{c.} \ldots\ldots$$

$$\therefore \quad y=\mathrm{P}_{\frac{1}{2}}\,\frac{\mathrm{D}}{\mathrm{D}-\frac{1}{2}}\,v$$

$$=\frac{\mathrm{D}\,(\mathrm{D}-\frac{1}{2})\,(\mathrm{D}-1)\ldots\ldots}{(\mathrm{D}-\frac{1}{2})\,(\mathrm{D}-1)\ldots\ldots}\,v$$

$$=\mathrm{D}\,v$$

$$=x\,\frac{dv}{dx}$$

Now $v+a\sqrt{-1}\,\frac{\overline{|-\mathrm{D}}}{\overline{|-\mathrm{D}+\frac{1}{2}}}\,e^{\frac{\theta}{2}}\,v=0$ is equivalent, by (D), to

$$v+a\sqrt{-1}\,e^{\frac{\theta}{2}}\,\frac{\overline{|-\mathrm{D}-\frac{1}{2}}}{\overline{|-\mathrm{D}}}\,v=0$$

or $$v-a\,\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\,v=0$$

whence $$v=\mathrm{A}_1\,e^{a^2 x}$$

$\therefore$ $$y=\mathrm{A}\,x\,e^{a^2 x} \text{ the same result as before.}$$

This process, which is due to Mr Boole, is of great importance in the solution of certain classes of ordinary linear equations, but I have not, as yet, found it very extensively applicable to equations with fractional indices.

Ex. 8. *More generally, to investigate the conditions of integrability of the equation*

$$x\frac{dy}{dx}-c\,y+a\,x^{n+\frac{1}{2}}\,\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0$$

The symbolical form is

$$\left(-\frac{\Gamma(-D+1)}{\Gamma(-D)}-c\right)y+a\sqrt{-1}\,e^{n\theta}\frac{\Gamma(-D+\frac{1}{2})}{\Gamma(-D)}\cdot y=0,$$

or $$-(-D+c)y+a\sqrt{-1}\frac{\Gamma(-D+n+\frac{1}{2})}{\Gamma(-D+n)}\cdot e^{n\theta}y=0, \text{ by (D).}$$

This is reducible,

1. When $c=n-\frac{1}{2}$; and it becomes, by dividing, by $-D+n-\frac{1}{2}$,

$$y-a\sqrt{-1}\frac{\Gamma(-D+n-\frac{1}{2})}{\Gamma(-D+n)}\cdot e^{n\theta}y=0$$

or $$a\sqrt{-1}\,e^{n\theta}y-\frac{\Gamma(-D+n)}{\Gamma(-D+n-\frac{1}{2})}\cdot y=0$$

or $$a\sqrt{-1}\,y-e^{-\frac{\theta}{2}}\frac{\Gamma(-D+\frac{1}{2})}{\Gamma(-D)}\cdot e^{-(n-\frac{1}{2})\theta}y=0$$

or $$ay+\frac{d^{\frac{1}{2}}\,y\,x^{-(n-\frac{1}{2})}}{dx^{\frac{1}{2}}}=0$$

If $yx^{-(n-\frac{1}{2})}=v$, this equation becomes

$$avx^{n-\frac{1}{2}}+\frac{d^{\frac{1}{2}}v}{dx^{\frac{1}{2}}}=0, \text{ or}$$

$$av+x^{-n+\frac{1}{2}}\frac{dv}{dx}=0$$

which is integrable when $n=\frac{1}{2}, 0, -\frac{1}{2}, -1,$ &c. (Class. 2.)

2. When $c=n$, the equation becomes

$$-y+a\sqrt{-1}\frac{\Gamma(-D+n+\frac{1}{2})}{\Gamma(-D+n+1)}\cdot e^{n\theta}y=0$$

or $$a\sqrt{-1}\,e^{n\theta}y-\frac{\Gamma(-D+n+1)}{\Gamma(-D+n+\frac{1}{2})}\cdot y=0$$

or $$a\sqrt{-1}\,y-e^{\frac{\theta}{2}}\frac{\Gamma(-D+\frac{1}{2})}{\Gamma(-D)}\cdot e^{-(n+\frac{1}{2})\theta}y=0$$

or $$ay+x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{y}{x^{n+\frac{1}{2}}}=0.$$

If $\frac{y}{x^{n+\frac{1}{2}}}=v$, this equation becomes

$$av+x^{-n+\frac{1}{2}}\frac{dv}{dx}=0; \text{ the same as before.}$$

Ex. 9. $$x\frac{dy}{dx}-ax^{3}\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}}-\frac{3}{2}y=0.$$

The symbolical form is

$$\frac{\lfloor -D+1}{\lfloor -D}\,y+a\,(-1)^{\frac{3}{2}}\,x^{\frac{3}{2}}\,\frac{\lfloor -D+\frac{3}{2}}{\lfloor -D}\,y+\tfrac{3}{2}\,y=0,$$

or $$(-D+\tfrac{3}{2})\,y+a\,(-1)^{\frac{3}{2}}\,\frac{\lfloor -D+3}{\lfloor -D+\frac{3}{2}}\,e^{\frac{3\theta}{2}}\,y=0,$$

or $$y+a\,(-1)^{\frac{3}{2}}\,\frac{\lfloor -D+3}{\lfloor -D+\frac{5}{2}}\,e^{\frac{3\theta}{2}}\,y=0,$$

or $$y+a\,(-1)^{\frac{3}{2}}\,e^{\frac{5\theta}{2}}\,\frac{\lfloor -D+\frac{1}{2}}{\lfloor -D}\,\frac{y}{e^{\theta}}=0$$

or $$y-a\,e^{2\theta}\,\frac{d^{\frac{1}{2}}\frac{y}{x}}{d\,x^{\frac{1}{2}}}=0$$

or $$\frac{y}{x}-a\,x\,\frac{d^{\frac{1}{2}}\frac{y}{x}}{d\,x^{\frac{1}{2}}}=0$$

$$y=\mathrm{A}\,\sqrt{x}\,e^{-\frac{1}{a^2 x}}\left\{\frac{1}{2}\int\frac{e^{\frac{1}{a^2 x}}}{\sqrt{x}}\,d\,x-\frac{\sqrt{\pi}\,\sqrt{-1}}{a}\right\}\qquad \text{(Ex. 3, Class 2.)}$$

Ex. 10. $$y+a\,x\,\frac{d\,y}{d\,x}+b\,x^{n}\left(\frac{d^{\frac{1}{2}}y}{d\,x^{\frac{1}{2}}}+c\,x\,\frac{d^{\frac{3}{2}}y}{d\,x^{\frac{3}{2}}}\right)+0.$$

The symbolical form of the equation is

$$y-a\,\frac{\lfloor -D+1}{\lfloor -D}\,y+b\,e^{(n-\frac{1}{2})\theta}\sqrt{-1}\left(\frac{\lfloor -D+\frac{1}{2}}{\lfloor -D}-c\,\frac{\lfloor -D+\frac{3}{2}}{\lfloor -D}\right)\cdot y=0$$

or $$(1+a\,D)\,y-b\sqrt{-1}\,e^{(n-\frac{1}{2})\theta}\,\frac{\lfloor -D+\frac{1}{2}}{\lfloor -D}\left(1+c\,D-\frac{c}{2}\right)\cdot y=0\quad (1).$$

This equation may be reduced in several instances:

A. If $c=\frac{2}{3}$ the equation becomes

$$(1+a\,D)\,y-b\sqrt{-1}\,\frac{2}{3}\,e^{(n-\frac{1}{2})\theta}\,\frac{\lfloor -D+\frac{1}{2}}{\lfloor -D}\cdot y=0,$$

or $$(1+a\,D)\,y-\frac{2}{3}\,b\sqrt{-1}\;\frac{\lfloor -D+n}{\lfloor -D+n-\frac{3}{2}}\,e^{(n-\frac{1}{2})\theta}\,y=0\ \text{ by (D.)}$$

or $$y+\frac{2\,b}{3\,a}\sqrt{-1}\,\frac{\lfloor -D+n}{(-D-\frac{1}{a})\,\lfloor -D+n-\frac{3}{2}}\cdot e^{(n-\frac{1}{2})\theta}\,y=0.\quad (2.)$$

1. If $\frac{1}{a}=1-n$, equation (2) is reduced to

$$y+\frac{2\,b}{3\,a}\sqrt{-1}\,\frac{\lfloor -D+n-1}{\lfloor -D+n-\frac{3}{2}}\cdot e^{(n-\frac{1}{2})\theta}\,y=0;$$

which is equivalent to $y+\frac{2b}{3a}\sqrt{-1}\,e^{(n-\frac{3}{2})\theta}\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\cdot xy=0$

or $$xy+\frac{2b}{3a}x^n\frac{d^{\frac{1}{2}}xy}{dx^{\frac{1}{2}}}=0\text{, or, if } v=xy,$$

$$v+\frac{2b}{3a}x^n\frac{d^{\frac{1}{2}}v}{dx^{\frac{1}{2}}}=0\text{, which is the form integrated in Class 1.}$$

2. If $\frac{1}{a}=\frac{3}{2}-n$, equation (2) becomes

$$y+\frac{2b}{3a}\sqrt{-1}\frac{\overline{|-D+n}}{\overline{|-D+n-\frac{1}{2}}}\cdot e^{(n-\frac{1}{2})\theta}y=0$$

which is equivalent to $y+\frac{2b\sqrt{-1}}{3a}e^{(n-\frac{1}{2})\theta}\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\cdot y=0$

or $$y+\frac{2b}{3a}x^n\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0$$

which is of the same form as in the last case.

B. If c is not equal to $\frac{2}{3}$, we have from equation (1) by (D.)

$$(1+aD)y+b\sqrt{-1}\frac{\overline{|-D+n}\,(1+cD-cn)}{\overline{|-D+n-\frac{1}{2}}}\cdot e^{(n-\frac{1}{2})\theta}y=0$$

$$y+b\sqrt{-1}\frac{1+cD-cn}{1+aD}\frac{\overline{|-D+n}}{\overline{|-D+n-\frac{1}{2}}}e^{(n-\frac{1}{2})\theta}y=0$$

3. If $\frac{c}{1-cn}=a$ this gives

$$y+b\sqrt{-1}\,(1-cn)\,e^{(n-\frac{1}{2})\theta}\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\cdot y=0$$

or $y+b\,(1-cn)\,x^n\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}=0$, the same form as before.

16. It would be improper to dismiss this equation without remarking the fact that it would appear to have been solved by M. Besge in Liouville's *Journal* 1844, ix., 294. The solution is, however, given without any demonstration, and is, if I mistake not, rather a differential equation *formed* than a differential equation *solved*. The *whole* which appears is as follows :

"Let m, n, p, q be functions of x, and $\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}}+m\frac{dy}{dx}+n\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+py=q$, the proposed equation.

"If we have $\frac{dm}{dx}+mn-p=0$, the given equation can be reduced to the following, $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+my=z$, where z is obtained from the equation $\frac{dz}{dx}+nz=q$."

Now, on examination, it appears that the proposed equation is nothing more than the differential coefficient of the quantity $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+my-z=0$ added to n times the quantity itself: Thus,

$$\frac{d}{dx}\left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+my-z\right)+n\left(\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+my-z\right)=0$$

gives

$$\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}}+m\frac{dy}{dx}+y\frac{dm}{dx}-\frac{dz}{dx}+n\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+mny-nz=0$$

or

$$\frac{d^{\frac{3}{2}}y}{dx^{\frac{3}{2}}}+m\frac{dy}{dx}+n\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+py=\frac{dz}{dx}+nz-\left(\frac{dm}{dx}+mn-p\right)y=q$$

provided $\frac{dz}{dx}+nz=q$ and $\frac{dm}{dx}+mn-p=0$.

Thus it appears that the equation is not *solved* but *formed:* and this is probably all M. BESGE intends. How he can justify his additional remark, that $\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+my=z$ can be solved if m is a constant, or a linear function of x, I am unable to conjecture.

CLASS 4. *Equations which are capable of solution by the division of operations.*

17. We have already met with several equations in Class 1, where the total operation was found to be equivalent to the product of two or more partial operations; and in Art. 9 we have pointed out the manner in which the partial operations are applied, viz., by decomposing the total operation in exactly the same way as an ordinary fraction is decomposed into partial fractions.

Ex. 1. $$y+axy+bx\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+2ax^2\frac{dy}{dx}=0.$$

This equation, when reduced to the symbolical form, is

$$y+ae^{\theta}y+be^{\frac{\theta}{2}}\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+\frac{1}{2}}}{\overline{|-\mathrm{D}}}\,y-2ae^{\theta}\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}}}\,y=0$$

or

$$y+b\sqrt{-1}\,\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}+\frac{1}{2}}}\,.\,e^{\frac{\theta}{2}}y-2a\left(\frac{\overline{|-\mathrm{D}+2}}{\overline{|-\mathrm{D}+1}}-\tfrac{1}{2}\right).\,e^{\theta}y=0 \text{ by (D).}$$

Now $$\frac{\overline{|-\mathrm{D}+2}}{\overline{|-\mathrm{D}+1}}-\tfrac{1}{2}=-\mathrm{D}+\tfrac{1}{2}=\frac{\overline{|-\mathrm{D}+\frac{3}{2}}}{\overline{|-\mathrm{D}+\frac{1}{2}}}=\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}+\frac{1}{2}}}\,.\,\frac{\overline{|-\mathrm{D}+\frac{3}{2}}}{\overline{|-\mathrm{D}+1}}$$

$$\therefore\quad\left(\frac{\overline{|-\mathrm{D}+2}}{\overline{|-\mathrm{D}+1}}-\tfrac{1}{2}\right)e^{\theta}y=\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}+\frac{1}{2}}}\,e^{\frac{\theta}{2}}\,\frac{\overline{|-\mathrm{D}+1}}{\overline{|-\mathrm{D}+\frac{1}{2}}}\,.\,e^{\frac{\theta}{2}}y \text{ by (D)}$$

and the equation is reduced to

$$y+b\sqrt{-1}\,\frac{\overline{|-D+1}}{\overline{|-D+\frac{1}{2}}}\,.\,\overset{\prime}{e^{z}}\,y-2\,a\,\frac{\overline{|-D+1}}{\overline{|-D+\frac{1}{2}}}\,\overset{\prime}{e^{z}}\,\frac{\overline{|-D+1}}{\overline{|-D+\frac{1}{2}}}\,\overset{\prime}{e^{z}}\,.\,y=0.$$

Let us abbreviate the operation $\frac{\overline{|-D+1}}{\overline{|-D+\frac{1}{2}}}\,\overset{\prime}{e^{z}}$ by ϕ, and the equation becomes

$$(1+b\sqrt{-1}\,\phi-2\,a\,\phi^2)\,.\,y=0.$$

If $1+b\sqrt{-1}\,z-2\,a\,z^2=(1+\alpha\,z)\,(1+\beta\,z)$; this equation is equivalent to

$$(1+\alpha\,\phi)\,(1+\beta\,\phi)y=0$$

or $$y=\frac{1}{(1+\alpha\,\phi)\,(1+\beta\,\phi)}\,.\,0=\frac{\alpha}{\alpha-\beta}\cdot\frac{1}{1+\alpha\,\phi}\,0-\frac{\beta}{\alpha-\beta}\,\frac{1}{1+\beta\,\phi}\,.\,0$$

Now $$1+\alpha\,\phi=1-\alpha\,\frac{\overline{|-D+1}}{\overline{|-D+\frac{1}{2}}}\,\overset{\prime}{e^{z}}=1+\alpha\,\overset{\prime}{e^{z}}\,\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}$$

$$=1-\alpha\sqrt{-1}\,x\,\frac{d^{\frac{1}{2}}}{d\,x^{\frac{1}{2}}}$$

Hence the solution of the given equation is reduced to the solution of the two equations

$$y_1+\alpha\,\overset{\prime}{e^{z}}\,\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\,y_1=0,\quad y_2+\beta\,\overset{\prime}{e^{z}}\,\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\,y_2=0$$

or $$y_1-\alpha\sqrt{-1}\,x\,\frac{d^{\frac{1}{2}}y_1}{d\,x^{\frac{1}{2}}}=0,\quad y_2-\beta\sqrt{-1}\,x\,\frac{d^{\frac{1}{2}}y_2}{d\,x^{\frac{1}{2}}}=0$$

Now these equations have been solved in Class 2, Ex. 3, and they give

$$y_1=\frac{A\,e^{\frac{1}{\alpha^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}}\,d\,x-\frac{\sqrt{\pi}}{\alpha}\right\}$$

$$y_2=\frac{B\,e^{\frac{1}{\beta^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}}\,d\,x-\frac{\sqrt{\pi}}{\beta}\right\}$$

and $$y=\frac{1}{(1-\alpha\,\phi)\,(1-\beta\,\phi)}\cdot 0=\frac{\alpha}{\alpha-\beta}\,\frac{1}{1-\alpha\,\phi}\cdot 0-\frac{\beta}{\alpha-\beta}\,\frac{1}{1-\beta\,\phi}\cdot 0$$

$$=\frac{A\,\alpha}{\alpha-\beta}\,\frac{e^{\frac{1}{\alpha^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}}\,d\,x-\frac{\sqrt{\pi}}{\alpha}\right\}$$

$$-\frac{B\,\beta}{\alpha-\beta}\,\frac{e^{\frac{1}{\beta^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}}\,d\,x-\frac{\sqrt{\pi}}{\beta}\right\}$$

It will be readily seen that B is not an arbitrary constant, independent of A.

(See Art. 18.) For when $b=0$, the equation becomes an ordinary linear equation of the first degree, of which the solution is $y=C\frac{e^{\frac{1}{2ax}}}{\sqrt{x}}$.

In this case $a=\beta$ and $A=-B$:

we may therefore write $B=-A$ generally, and we obtain as the complete solution

$$y=\frac{A\,e^{\frac{1}{\alpha^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}}\,dx-\frac{\sqrt{\pi}}{\alpha}\right\}-\frac{A\,e^{\frac{1}{\beta^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}}\,dx-\frac{\sqrt{\pi}}{\beta}\right\}$$

The above equation may be reduced differently, thus. The symbolical form

$$y+a e^{\theta}y+b\,e^{\frac{\theta}{2}}\sqrt{-1}\,\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\,y-2a e^{\theta}\,\frac{\overline{|-D+1}}{\overline{|-D}}\,y=0,$$

may be written

$$2a(-D-\tfrac{1}{2})y-b\sqrt{-1}\,e^{-\frac{\theta}{2}}\frac{\overline{|-D+\frac{1}{2}}}{\overline{|-D}}\,y-\frac{1}{2a}e^{-\theta}y=0,$$

or
$$y-\frac{b\sqrt{-1}}{2a}\,\frac{\overline{|-D}}{\overline{|-D+\frac{1}{2}}}\,e^{-\frac{\theta}{2}}y-\frac{1}{2a}\,\frac{1}{-D-\frac{1}{2}}\,e^{-\theta}y=0,$$

or
$$y-\frac{b\sqrt{-1}}{2a}\,\frac{\overline{|-D}}{\overline{|-D+\frac{1}{2}}}\,e^{-\frac{\theta}{2}}y-\frac{1}{2a}\,\frac{\overline{|-D}}{\overline{|-D+\frac{1}{2}}}\,\frac{\overline{|-D-\frac{1}{2}}}{\overline{|-D}}\,e^{-\theta}y=0,$$

or
$$y-\frac{b\sqrt{-1}}{2a}\,\frac{\overline{|-D}}{\overline{|-D+\frac{1}{2}}}\,e^{-\frac{\theta}{2}}y-\frac{1}{2a}\,\frac{\overline{|-D}}{\overline{|-D+\frac{1}{2}}}\,e^{-\frac{\theta}{2}}\,\frac{\overline{|-D}}{\overline{|-D+\frac{1}{2}}}\,e^{-\frac{\theta}{2}}y=0,$$

which is of the form $(1-\frac{b\sqrt{-1}}{2a}\phi_1-\frac{1}{2a}\phi_1{}^2)y=0$;

of which the solutions are

$$(1+\frac{1}{a}\phi_1)\,y=0,\ \text{and}\ (1+\frac{1}{\beta}\phi_1)\,y=0,\ \text{or}$$

$$y+\frac{1}{a}\,e^{\frac{\theta}{2}}\,\frac{\overline{|-D-\frac{1}{2}}}{\overline{|-D}}\,e^{-\theta}y=0,\ \text{and}\ y+\frac{1}{\beta}\,e^{\frac{\theta}{2}}\,\frac{\overline{|-D-\frac{1}{2}}}{\overline{|-D}}\,e^{-\theta}y=0,$$

or
$$y-\frac{1}{a\sqrt{-1}}\,\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\,\frac{y}{x}=0,\ \text{and}\ y-\frac{1}{\beta\sqrt{-1}}\,\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\,\frac{y}{x}=0;$$

which, on differentiation to the index $\frac{1}{2}$, give the same results as before.

Ex. 2. $$y+axy+bx\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+2ax^2\frac{dy}{dx}=X.$$

The solution is, as in Example 1,

$$y=\frac{a}{a-\beta}\,\frac{1}{1+a\phi}(X+0)-\frac{\beta}{a-\beta}\,\frac{1}{1+\beta\phi}(X+0)$$

Now $\frac{X+0}{1+a\phi}$ is the solution of the equation $y_1 - a\sqrt{-1}\, x \frac{d^{\frac{1}{2}} y_1}{dx^{\frac{1}{2}}} = X + 0$; which (Class II, Ex. 8, Cor. 1) is

$$\left(1 + a\sqrt{-1}\, x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\right) \left(\frac{A e^{\frac{1}{a^2 x}}}{\sqrt{x}} + \frac{e^{\frac{1}{a^2 x}}}{\sqrt{x}} \int \frac{e^{-\frac{1}{a^2 x}}}{a^2 x^{\frac{3}{2}}} X\, dx\right)$$

and a similar equation results for β. Hence the solution of the given equation is known.

18. It must be remarked of this solution, that it is not in all cases complete without the introduction of the complementary (or arbitrary) function. This arises from the circumstance that when y contains positive integral powers of x, $\frac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}} = 0$, whereas $x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} x \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} y$ is not equal to 0.

Hence $x^2 \frac{dy}{dx} + \frac{1}{2} x y$ can be replaced by the latter function only by the convention that $\frac{d^{\frac{1}{2}} x^n}{dx^{\frac{1}{2}}}$ is not to be written 0 when n is a positive integer.

On account of this convention, the solution of the equation $\frac{1}{1+a\phi} y = X$ must contain, besides the expression given for it above, a series of positive integral powers of x; and hence y, the solution of Equation (2), is incomplete without the addition of such a function. It is probable, however, that the determination of a relation between the arbitrary constants may give a solution possessing all the generality which the science is capable of. We have already given an example of the mode of avoiding arbitrary functions by introducing such a relation in Example 1. We shall offer another as a corollary.

Cor. If $X = \frac{e}{\sqrt{x}}$, the solution is (Class 2. Ex. 8, Cor 2.)

$$y = y_0 + \frac{a}{a-\beta}\left(\frac{e}{\sqrt{x}} - \frac{a e}{\sqrt{x}}\right) - \frac{\beta}{a-\beta}\left(\frac{e}{\sqrt{x}} - \frac{\beta e}{\sqrt{x}}\right) + \text{arbitrary function}$$

$$= y_0 + \frac{e}{\sqrt{x}} - (a+\beta)\frac{e}{\sqrt{\pi}} + \text{arbitrary function}$$

$$= y_0 + \frac{e}{\sqrt{x}} - (a+\beta)\frac{e}{\sqrt{\pi}} + p x + q x^2 + \&c.$$

Now if we examine the equation which connects together p, q, &c., we shall find that it is the same as that which determines y_1 in Class 2, Ex. 3, having only $2a$ in place of a^2. Hence it is contained in the solution of the given equation when b and X are omitted. It is, therefore, itself only a supplementary term in the solution of the given equation, and its place may be supplied, appa-

rently without any sacrifice of generality, by the introduction of a relation between A and B. The relation is, $A+B=\frac{be\sqrt{-1}}{\sqrt{\pi}}$.

Hence the complete solution of the equation

$$y+axy+bx\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}+2ax^2\frac{dy}{dx}=\frac{e}{\sqrt{x}}, \text{ is}$$

$$y=\frac{Ae^{\frac{1}{\alpha^2 x}}}{\sqrt{x}}\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\alpha^2 x}}}{\sqrt{x}}dx-\frac{\sqrt{\pi}}{\alpha}\right\}-\left(A-\frac{be\sqrt{-1}}{\sqrt{\pi}}\right)\left\{\frac{1}{2}\int\frac{e^{-\frac{1}{\beta^2 x}}}{\sqrt{x}}dx\right.$$
$$\left.-\frac{\sqrt{\pi}}{\beta}\right\}+\frac{e}{\sqrt{x}}-\frac{be\sqrt{-1}}{\sqrt{\pi}}$$

19. Ex. 3. $$xy+ax\frac{d^{-1}y}{dx^{-1}}-(a+bx)\frac{d^{-2}y}{dx^{-2}}+2b\frac{d^{-3}y}{dx^{-3}}=0.$$

Multiply by x^{-3}, and the result will be

$$x^{-2}y+ax^{-2}\frac{d^{-1}y}{dx^{-1}}-(ax^{-3}+bx^{-2})\frac{d^{-2}y}{dx^{-2}}+2bx^{-3}\frac{d^{-3}y}{dx^{-3}}=0$$

of which the symbolical form is

$$ye^{-2\theta}-ae^{-\theta}\frac{\lfloor\overline{-D-1}}{\lfloor\overline{-D}}y-(ae^{-\theta}+b)\frac{\lfloor\overline{-D-2}}{\lfloor\overline{-D}}y-2b\frac{\lfloor\overline{-D-3}}{\lfloor\overline{-D}}y=0$$

which is equivalent to

$$ye^{-2\theta}+ae^{-\theta}\left(\frac{1}{D+1}-\frac{1}{(D+1)(D+2)}\right)y-b\left(\frac{1}{(D+1)(D+2)}-\frac{2}{(D+1)(D+2)(D+3)}\right)y=0$$

or $$ye^{-2\theta}+ae^{-\theta}\frac{1}{D+2}y-b\frac{1}{(D+2)(D+3)}y=0$$

Hence, by multiplication,

$$y-\frac{a}{b}(D+2)(D+3)e^{-\theta}\frac{1}{D+2}y-\frac{1}{b}(D+2)(D+3)e^{-2\theta}y=0$$

or $$y-\frac{a}{b}(D+2)e^{-\theta}y-\frac{1}{b}(D+2)e^{-\theta}(D+2)e^{-\theta}y=0 \text{ by (B)}$$

which is of the form $$\left(1-\frac{a}{b}\phi-\frac{1}{b}\phi^2\right)y=0;$$

which, being put under the form

$$(1+\alpha\phi)(1+\beta\phi)y=0 \text{ gives}$$
$$y=A(1+\alpha\phi)^{-1}.0+B(1+\beta\phi)^{-1}.0$$

Now $(1+\alpha\phi)^{-1}.0$ is the solution of the equation

$$y_1+\alpha(D+2)y_1e^{-\theta}=0 \text{ or } y_1+\frac{\alpha}{x}\frac{dxy_1}{dx}=0$$

of which the result is $y_1=\frac{A_1}{x}e^{-\frac{x}{\alpha}}$.

Hence $y=\frac{A}{x}e^{-\frac{x}{\alpha}}+\frac{B}{x}e^{-\frac{x}{\beta}}$ is the complete solution of the given equation.

In my second Memoir on this subject, I exemplified the use of a theorem in general differentiation, by solving the problem of determining the law of force by which the particles of a sphere must act on a point, so that the whole attraction may be the same as if the sphere were collected at its centre of gravity. The solution of this problem led to a differential equation which was shewn, by an indirect process, to be satisfied by the law of force varying as the distance, or inversely as its square. I propose, at present, to solve this differential equation.

Ex. 4. The equation is (vol. xiv., p. 608).

$$\frac{\pi}{4a^2}\left\{8aR(a+R)\frac{d^{-2}y}{dz^{-2}}-8(a^2+3aR+R^2)\frac{d^{-3}y}{dz^{-3}}\right.$$

$$\left.+24(a+R)\frac{d^{-4}y}{dz^{-4}}-24\frac{d^{-5}y}{dz^{-5}}\right\}=\frac{4\pi}{3}R^3f(a)$$

where $y=f(z+\alpha)$, $z=2R$, $\alpha=a-R$.

This becomes, by substitution,

$$4az\left(a+\frac{z}{2}\right)\frac{d^{-2}y}{dz^{-2}}-8\left(a^2+\frac{3az}{2}+\frac{z^2}{4}\right)\frac{d^{-3}y}{dz^{-3}}$$

$$+24\left(a+\frac{z}{2}\right)\frac{d^{-4}y}{dz^{-4}}-24\frac{d^{-5}y}{dz^{-5}}=\frac{2}{3}a^2faz^3.$$

Dividing by z^5, we get

$$\frac{4a^2}{z^4}\frac{d^{-2}y}{dz^{-2}}+\frac{2a}{z^3}\frac{d^{-2}y}{dz^{-2}}-\frac{8a^2}{z^5}\frac{d^{-3}y}{dz^{-3}}-\frac{12a}{z^4}\frac{d^{-3}y}{dz^{-3}}-\frac{2}{z^3}\frac{d^{-3}y}{dz^{-3}}$$

$$+\frac{24a}{z^5}\frac{d^{-4}y}{dz^{-4}}+\frac{12}{z^4}\frac{d^{-4}y}{dz^{-4}}-\frac{24}{z^5}\frac{d^{-5}y}{dz^{-5}}=\frac{2}{3}\frac{a^2f(a)}{z^2}$$

Writing e^θ for z, and $(-1)^{-\mu}\frac{\overline{|-D-\mu}}{\overline{|-D}}y$ for $\frac{1}{z^\mu}\frac{d^{-\mu}y}{dz^{-\mu}}$, there results the symbolical form

$$4a^2e^{-2\theta}\frac{\overline{|-D-2}}{\overline{|-D}}y+2ae^{-\theta}\frac{\overline{|-D-2}}{\overline{|-D}}y+8a^2e^{-2\theta}\frac{\overline{|-D-3}}{\overline{|-D}}y$$

$$+12ae^{-\theta}\frac{\overline{|-D-3}}{\overline{|-D}}y+2\frac{\overline{|-D-3}}{\overline{|-D}}y+24ae^{-\theta}\frac{\overline{|-D-4}}{\overline{|-D}}y$$

$$+12\frac{\overline{|-D-4}}{\overline{|-D}}y+24\frac{\overline{|-D-5}}{\overline{|-D}}y=\frac{2}{3}a^2f(a)e^{-2\theta};$$

or, collecting the terms,

$$\left\{24\frac{\overline{|-D-5}}{\overline{|-D}}+12\frac{\overline{|-D-4}}{\overline{|-D}}+\frac{\overline{|-D-3}}{\overline{|-D}}\right\}y+$$

$$+ae^{-\theta}\left\{24\frac{|\overline{-D-4}}{|\underline{-D}}+12\frac{|\overline{-D-3}}{|\underline{-D}}+2\frac{|\overline{-D-2}}{|\underline{-D}}\right\}y$$

$$+a^2e^{-2\theta}\left\{8\frac{|\overline{-D-3}}{|\underline{-D}}+4\frac{|\overline{-D-2}}{|\underline{-D}}\right\}y=\frac{2}{3}a^2f(a)e^{-2\theta}$$

which being reduced gives

$$2(D+1)(D+2)\frac{|\overline{-D-5}}{|\underline{-D}}y+2ae^{-\theta}D(D+1)\frac{|\overline{-D-4}}{|\underline{-D}}y$$

$$-4a^2e^{-2\theta}(D+1)\frac{|\overline{-D-3}}{|\underline{-D}}y=\frac{2}{3}a^2f(a)e^{-2\theta};$$

or

$$-\frac{1}{(D+3)(D+4)(D+5)}y+ae^{-\theta}\frac{D}{(D+2)(D+3)(D+4)}y$$

$$+2a^2e^{-2\theta}\frac{1}{(D+2)(D+3)}y=\frac{1}{3}a^2f(a)e^{-2\theta}$$

Now $y=f(a+z)$, but since a is itself a function of z, we cannot proceed further with the reduction of this equation by division, but must proceed to obtain a relation between a and R or a and z.

To do this we shall expand $f(a+z)$ by TAYLOR's Theorem.

The result is

$y=\Sigma\frac{d^nf(\alpha)}{d\alpha^n}\frac{z^n}{|\underline{n+1}}$ which being substituted in the reduced equation, gives by (A)

$$\Sigma\left(\frac{d^nf(\alpha)}{d\alpha^n}\frac{e^{n\theta}}{|\underline{n+1}}\right)\left\{\frac{-1}{(n+3)(n+4)(n+5)}\right.$$

$$\left.+a\frac{n}{(n+2)(n+3)(n+4)}e^{-\theta}+\frac{2a^2e^{-2\theta}}{(n+2)(n+3)}\right\}=\frac{a^2}{3}fae^{-2\theta}$$

But $\alpha=a-R=a-\frac{z}{2}$: hence

$$\frac{d^nf(\alpha)}{d\alpha^n}=\frac{d^nfa}{da^n}-\frac{d^{n+1}fa}{da^{n+1}}\frac{z}{2}+\text{\&c.}$$

which being substituted for $\frac{d^nf(\alpha)}{d\alpha^n}$, the sum being taken for n and p, we get

$$\Sigma_{n,p}\frac{(-1)^p}{2^p}\frac{d^{n+p}fa}{da^{n+p}}\frac{1}{|\underline{n+1}\,|\underline{p+1}}\left\{\frac{-z^{n+p}}{(n+3)(n+4)(n+5)}\right.$$

$$\left.+\frac{anz^{n+p-1}}{(n+2)(n+3)(n+4)}+\frac{2a^2z^{n+p-2}}{(n+2)(n+3)}\right\}=\frac{a^2}{3}\frac{fa}{z^2}$$

every integer value of n and p being taken from 0 to ∞.

When $n=0$, $p=0$, the left-hand side gives

$$\frac{-fa}{60}+\frac{a^2fa}{3z^2}$$

$$n=0, p=1, \ldots \frac{z}{120}\frac{d f a}{d a}-\frac{a^2}{6z}\frac{d f a}{d a}$$

$$n=0, p=2, \ldots \frac{-z^2}{480}\frac{d^2 f a}{d a^2}+\frac{a^2}{24}\frac{d^2 f a}{d a^2}$$

When $$n=1, p=0, \ldots \frac{-z}{120}\frac{d f a}{d a}+\frac{a}{60}\frac{d f a}{d a}$$

$$+\frac{a^2}{6z}\frac{d f a}{d a}$$

$$n=1, p=1, \frac{z^2}{240}\frac{d^2 f a}{d a^2}-\frac{a z}{120}\frac{d^2 f a}{d a^2}-\frac{a^2}{12}\frac{d^2 f a}{d a^2}$$

$$n=2, p=0, \frac{-z^2}{420}\frac{d^2 f a}{d a^2}+\frac{a z}{120}\frac{d^2 f a}{d a^2}+\frac{a^2}{20}\frac{d^2 f a}{d a^2}, \text{\&c., \&c.}$$

Hence we obtain, by collecting the terms and equating their sum to $\frac{a^2}{3}\frac{f a}{z^2}$,

$$\frac{a^2 f a}{3 z^2}-\frac{f a}{60}+\frac{a}{60}\frac{d f a}{d a}+\frac{a^2}{120}\frac{d^2 f a}{d a^2}+\mathrm{P} z+\mathrm{Q} z^2+\text{\&c.}$$

$$=\frac{a^2}{3}\frac{f a}{z^2}$$

Equating coefficients of like powers of z, we obtain

$$-\frac{f a}{60}+\frac{a}{60}\frac{d f a}{d a}+\frac{a^2}{120}\frac{d^2 f a}{d a^2}=0.$$

This equation will determine $f(a)$, the only law of force by which a sphere can attract an external particle exactly as much as if it were all collected at its centre of gravity.

The symbolical form of the equation is

$$\{\mathrm{D}(\mathrm{D}-1)+2\mathrm{D}-2\} f a=0$$

or $$(\mathrm{D}^2+\mathrm{D}-2) f a=0, \text{ or } (\mathrm{D}-1)(\mathrm{D}+2) f a=0.$$

Hence $$(\mathrm{D}-1) f a=0, (\mathrm{D}+2) f a=0,$$

or $$\frac{d f a}{d a}=f a, \text{ and } \frac{d f a}{d a}=-2 f a$$

that is $f a=\mathrm{A}\, a$, and $f a=\frac{\mathrm{B}}{a^2}$ are particular integrals, and the complete integral is $y=\mathrm{A}\, a+\frac{\mathrm{B}}{a^2}$; which is the law required.

Section II. Simultaneous Equations.

20. To effect the solution of simultaneous equations, we must eliminate one of the quantities differentiated. This is best effected by treating both the differ-

entiation and the multiplication by a constant in the same manner, regarding both the one and the other as an operation. A similar process has been employed for the solution of ordinary simultaneous equations by Mr GREGORY in the *Cambridge Mathematical Journal*, i. 173.

Ex. 1. $$\frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + a y = 0, \quad \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + b x = 0$$

By taking the $\frac{1}{2}$ differential of the first equation, we get $\frac{d x}{d t} + a \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} = 0$; which, by virtue of the second, gives

$$\frac{d x}{d t} - a b x = 0 \text{; or } x = \mathrm{A}\, e^{a b t} \qquad \therefore y = -\mathrm{A} \sqrt{\frac{b}{a}}\, e^{a b t}.$$

Ex. 2. $$\frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + a y + b x = 0, \quad \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + a' y + b' x = 0.$$

These equations may be written

$$\left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + b\right) x + a y = 0, \quad \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a'\right) y + b' x = 0 ;$$

whence, by eliminating y, we obtain

$$\left\{ \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a'\right) \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + b\right) - a b' \right\} x = 0.$$

Let $\alpha^{\frac{1}{2}}$, $\beta^{\frac{1}{2}}$, be the roots of the equation

$$(z + a')(z + b) - a b' = 0$$

then $$\left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} - \alpha^{\frac{1}{2}}\right) \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} - \beta^{\frac{1}{2}}\right) x = 0 \text{; which gives}$$

$$x = \mathrm{A}\, e^{\alpha t} + \mathrm{B}\, e^{\beta t}$$

and $$y = -\frac{1}{a} \frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} - \frac{b}{a} x$$

$$= -\left(\frac{\alpha^{\frac{1}{2}}}{a} + \frac{b}{a}\right) \mathrm{A}\, e^{\alpha t} - \left(\frac{\beta^{\frac{1}{2}}}{a} + \frac{b}{a}\right) \mathrm{B}\, e^{\beta t}.$$

Ex. 3. $$\frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + a y + b x = f(t)$$

$$\frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + a' y + b' x = \phi(t)$$

$$\left\{ \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a'\right) \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + b\right) - a b' \right\} x = \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a'\right) f(t) - a \phi t$$

$$= \psi(t)$$

This coincides with Ex. 4, Class 1, and the solution is

$$x = \mathrm{A} e^{\alpha t} + \mathrm{B}\, e^{\beta t} + \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} - \alpha^{\frac{1}{2}}\right)^{-1} \psi t - \frac{1}{\alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}}} \left(\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} - \beta^{\frac{1}{2}}\right)^{-1} \psi(t)$$

Ex. 4.
$$\frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + a x + b y + c z = 0$$
$$\frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + a' x + b' y + c' z = 0$$
$$\frac{d^{\frac{1}{2}} z}{d t^{\frac{1}{2}}} + a'' x + b'' y + c'' z = 0$$

These equations can be written in the form
$$\mathrm{A} x + b y + c z = 0$$
$$a' x + \mathrm{B}' y + c' z = 0$$
$$a'' x + b'' y + \mathrm{C}'' z = 0$$

By eliminating y and z we obtain
$$\{\mathrm{A}\,\mathrm{B}'\,\mathrm{C}'' - b'' c' \mathrm{A} - a'' c \mathrm{B}' - a' b \mathrm{C}'' + a' b'' c + a'' b c'\} x = 0$$

or
$$\left\{\frac{d^{\frac{3}{2}}}{d t^{\frac{3}{2}}} + (a + b' + c)\frac{d}{d t} + (a b' + a c'' + b c'')\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a b' c'' - b'' c' \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a b'' c' - a'' c \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a'' b' c - a' b \frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a' b c'' + a' b'' c + a'' b c'\right\} x = 0$$

or
$$\left\{\frac{d^{\frac{3}{2}}}{d t^{\frac{3}{2}}} + (a + b + c)\frac{d}{d t} + (a b' + a c'' + b'' c' - b c'' - a'' c' - a' b)\frac{d^{\frac{1}{2}}}{d t^{\frac{1}{2}}} + a b' c'' + a b'' c' + a'' b' c + a' b c'' + a' b'' c + a'' b c'\right\} x = 0$$

which is of the same form as Cor. 2, Ex. 3, Class 1, and its integral is therefore known.

Knowing x, we have $b y + c z = f(t)$ by the first of the three equations, and $\frac{b\, d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + c \frac{d^{\frac{1}{2}} z}{d t^{\frac{1}{2}}} = \varphi(t)$ by differentiation, whence, by substituting the values of $\frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}}$ and $\frac{d^{\frac{1}{2}} z}{d t^{\frac{1}{2}}}$ from the second and third equations, there results a second equation between y, z, and t. From these two equations y and z are determined in terms of t.

Ex. 5. Given
$$\frac{d x}{d t} + a \frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + b x = p \frac{d y}{d t} + q \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + r y$$
$$\frac{d x}{d t} + a' \frac{d^{\frac{1}{2}} x}{d t^{\frac{1}{2}}} + b' x = p' \frac{d y}{d t} + q' \frac{d^{\frac{1}{2}} y}{d t^{\frac{1}{2}}} + r' y$$

These equations may be written
$$(d + a\, d^{\frac{1}{2}} + b) x = (p\, d + q\, d^{\frac{1}{2}} + r) y$$
$$(d + a'\, d^{\frac{1}{2}} + b') x = (p'\, d + q'\, d^{\frac{1}{2}} + r') y$$

$$\therefore \quad (p'd+q'd^{\frac{1}{2}}+r')(d+a\,d^{\frac{1}{2}}+b)\,x=(p\,d+q\,d^{\frac{1}{2}}+r)(d+a'd^{\frac{1}{2}}+b')\,x$$

or

$$(p-p')\frac{d^2x}{dt^2}+(p\,a'+q-p'a-q')\frac{d^{\frac{3}{2}}x}{dt^{\frac{3}{2}}}+(p\,b'+q\,a'+r$$

$$-p'b-q'a-r')\frac{dx}{dt}+(q\,b'+r\,a'-q'b-r'a)\frac{d^{\frac{1}{2}}x}{dt^{\frac{1}{2}}}$$

$$+r\,b'-r'b)\,x=0$$

which coincides with the general form Cor. 2, Ex. 3, Class 1.

Section III. Partial Differential Equations.

21. In order to effect the solution of partial differential equations in which the operation with respect to x is totally independent of the operation with respect to y, we must distinguish the operation of differentiation in the two cases by different symbols. Let d stand for $\frac{d}{dx}$, δ for $\frac{d}{dy}$: then in solving the equation with respect to d, we may treat δ as a constant, and *vice versâ*.

Ex. 1. $$\frac{d^{\frac{1}{2}}z}{dx^{\frac{1}{2}}}-b\frac{d^{\frac{1}{2}}z}{dy^{\frac{1}{2}}}=0.$$

Write this equation $(d^{\frac{1}{2}}-b\,\delta^{\frac{1}{2}})\,z=0$: In this form it coincides with Ex. 1, Class 1, and its solution is $z=\mathrm{A}\,e^{b^2\delta x}$.

Now A is an arbitrary function of y; call it $f(y)$: then $z=e^{b^2x\delta}f(y)$, where $e^{b^2x\delta}$ represents an operation on $f(y)$.

But since
$$f(y+k)=fy+\frac{dfy}{dy}k+$$
$$=(1+k\,\delta+\&c.)f(y)$$
$$=e^{k\delta}f(y),$$
it is evident that
$$z=f(y+b^2x).$$

Ex. 2. $$\frac{d^{\frac{1}{2}}z}{dx^{\frac{1}{2}}}=a\frac{dz}{dy}.$$

This equation may be written $(d^{\frac{1}{2}}-a\,\delta)\,z=0$.

$$\therefore \quad z=e^{a^2x\delta^2}f(y).$$

Now $$\int_{-\infty}^{\infty}d\omega\,e^{-(\omega-b)^2}=\sqrt{\pi} \quad \text{(Gregory's \textit{Examples}, p. 499.)}$$

$$\therefore \quad z\sqrt{\pi}=\int_{-\infty}^{\infty}d\omega\,e^{-\omega^2+2b\omega-b^2}.\,e^{a^2x\delta^2}f(y)$$

$$=\int_{-\infty}^{\infty}d\omega\,e^{\omega^2}\,e^{2\omega a\sqrt{x}\delta}f(y) \quad (\text{if } b=a\sqrt{x}\,\delta)$$

$$=\int_{-\infty}^{\infty} d\omega\, e^{-\omega^2} f(y+2\omega a\sqrt{x})$$

which is the solution of the equation in the form of a definite integral.

Ex. 3. $$\frac{d^{\frac{1}{2}} z}{d x^{\frac{1}{2}}}=a\frac{dz}{dy}+cz$$

The first form of the solution is evidently

$$z=e^{(a\delta+c)^2 x} f(y)$$

which is reduced to $$z=e^{c^2 x} e^{a^2\delta^2 x} f(y+2acx)$$

$$=\frac{e^{c^2 x}}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-\omega^2} f(y+2acx+2\omega a\sqrt{x})$$

as in Example 2.

Ex. 4. $$\frac{dz}{dx}-2a\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\cdot\frac{d}{dy}z+a^2\frac{d^2 z}{dy^2}=c^2 z$$

This equation may be written $(d-2ad^{\frac{1}{2}}\delta+a^2\delta^2-c^2)z=0$, which is of the form of Ex. 3, Class 1, and the solution is

$$z=e^{c^2 x} e^{a^2 x \delta^2}\{e^{2acx\delta} f(y)+e^{-2acx\delta}\phi(y)\}$$

$$=e^{c^2 x} e^{a^2 x\delta^2}\{f(y+2acx)+\phi(y-2acx)\}$$

$$=\frac{e^{c^2 x}}{\sqrt{\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-\omega^2}\{f(y+2acx+2\omega a\sqrt{x})+\phi(y-2acx+2\omega a\sqrt{x})\}$$

Ex. 5. $$\frac{dz}{dx}-2a\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ \frac{d^{\frac{1}{2}}}{dy^{\frac{1}{2}}}\ z+a^2\frac{dz}{dy}=0.$$

This equation gives $(d-2ad^{\frac{1}{2}}\delta^{\frac{1}{2}}+a^2\delta)z=0$

or $$z=e^{a^2 x\delta} f(y)+x e^{a^2 x\delta}\phi(y) \quad \text{(Class 1, Ex. 3. Cor. 1.)}$$

$$=f(y+a^2 x)+x\phi(y+a^2 x)$$

Ex. 6. $$\frac{dz}{dx}+a\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\cdot\frac{d^{\frac{1}{2}}}{dy^{\frac{1}{2}}}z+b\frac{dz}{dy}=cz.$$

This equation may be written $(d+a\delta^{\frac{1}{2}}d^{\frac{1}{2}}+b\delta-c)z=0$ which coincides with Ex. 3, Class 1; and the solution is

$$z=e^{\alpha x} f(y)+e^{\beta x}\phi(y)$$

where $\alpha^{\frac{1}{2}}$, $\beta^{\frac{1}{2}}$ are the roots of the equation in v

$$v+a\delta^{\frac{1}{2}}v^{\frac{1}{2}}+b\delta-c=0:$$

or $$\alpha^{\frac{1}{2}}=-\frac{a\delta^{\frac{1}{2}}}{2}+\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}$$

$$\beta^{\frac{1}{2}}=-\frac{a\delta^{\frac{1}{2}}}{2}-\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}$$

or
$$\alpha=\left(\frac{a^2}{2}-b\right)\delta+c-a\,\delta^{\frac{1}{2}}\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}$$

$$\beta=\left(\frac{a^2}{2}-b\right)\delta+c+a\,\delta^{\frac{1}{2}}\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}$$

$$\therefore\quad z=e^{cx}\,e^{\left(\frac{a^2}{2}-b\right)\delta x}\left\{e^{-a\delta^{\frac{1}{2}}x\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}}f(y)+e^{a\delta^{\frac{1}{2}}x\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}}\phi(y)\right\}$$

$$=e^{cx}\left\{e^{-a\delta^{\frac{1}{2}}x\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}}f\left(y+\left(\frac{a^2}{2}-b\right)x\right)+e^{a\delta^{\frac{1}{2}}x\sqrt{\left(\frac{a^2}{4}-b\right)\delta+c}}\phi\left(y+\left(\frac{a^2}{2}-b\right)x\right)\right\}$$

These expressions do not appear to be susceptible of further reduction, except in particular cases.

COR. Let $b=\frac{a^2}{4}$; then

$$z=e^{cx}\left\{e^{-ax\delta^{\frac{1}{2}}\sqrt{c}}f\left(y+\frac{a^2}{4}x\right)+e^{ax\delta^{\frac{1}{2}}\sqrt{c}}\phi\left(y+\frac{a^2}{4}x\right)\right\}$$

To reduce this expression, it may perhaps be sufficiently general to suppose the symbol $\delta^{\frac{1}{2}}$ to include both the positive and negative signs, in which case we may write only one of the functions $e^{-a\sqrt{c}\,x\,\delta^{\frac{1}{2}}}f(y+\frac{a^2}{4}x)$.

Now $\qquad \frac{\sqrt{\pi}}{2}e^{-2v}=\int_0^\infty d\omega\, e^{-\left(\omega^2+\frac{v^2}{\omega^2}\right)}$. (See GREGORY'S *Examples*, p. 499.)

Let $v=\frac{ax\sqrt{c}\,\delta^{\frac{1}{2}}}{2}$; then

$$\frac{\sqrt{\pi}}{2}e^{-a\sqrt{c}\,x\,\delta^{\frac{1}{2}}}=\int_0^\infty d\omega\, e^{-\left(\omega^2+\frac{a^2cx^2\delta}{4\omega^2}\right)}$$

and
$$z=e^{cx}\,e^{-a\sqrt{c}\,x\,\delta^{\frac{1}{2}}}f\left(y+\frac{a^2}{4}x\right)$$

$$=e^{cx}\frac{2}{\sqrt{\pi}}\int_0^\infty d\omega\, e^{-\left(\omega^2+\frac{a^2cx^2\delta}{4\omega^2}\right)}f\left(y+\frac{a^2}{4}x\right)$$

$$=\frac{2}{\sqrt{\pi}}\,e^{cx}\int_0^\infty d\omega\, e^{-\omega^2}\,f\left(y+\frac{a^2}{4}x-\frac{a^2cx^2}{4\omega^2}\right).$$

SECTION IV. DIFFERENCES.

22. The definition of the difference of u_x, as it is commonly written by English authors is $u_{x+1}-u_x$. We shall retain this definition, and generalize it by writing $e^{\frac{d}{dx}}u_x$ for u_{x+1}, and consequently $(e^{\frac{d}{dx}}-1)\,u_x$ for Δu_x.

The results which we shall produce from this definition, as applied to frac-

tional values of the index of difference will, in most cases, differ not at all from the results obtained in the ordinary calculus of differences. We offer them only for the purpose of exhibiting those formulæ which possess all the generality which can be desired, at a single glance.

Suppose, then, $\Delta u_x = f(u_x) = v_x$

$$\therefore \quad \Delta^2 u_x = \Delta v_x = (e^{\frac{d}{dx}} - 1)\, v_x = (e^{\frac{d}{dx}} - 1)^2 u_x, \text{ \&c.}$$

and, according to the axiom of the calculus of operations that the repetitions of equivalent operations are equivalent, we shall have generally $\Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x$: whatever n may be. This, then, may be said to be the *definition* of $\Delta^n u_x$.

Also, since $u_{x+n} = e^{n\frac{d}{dx}} u_x$ by TAYLOR's Theorem, and $\Delta u_x = (e^{\frac{d}{dx}} - 1)\, u_x$; it follows that $u_{x+n} = (1+\Delta)^n u_x$.

We proceed now to apply it to the demonstration of the theorems which connect together $\Delta^r u_{x+s}$, and u_{x+p}, &c.

$$(1). \quad \Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x = \left(e^{n\frac{d}{dx}} - n\, e^{(n-1)\frac{d}{dx}} + \frac{n(n-1)}{1.2} e^{(n-2)\frac{d}{dx}} - \text{\&c.}\right) u_x$$

$$= u_{x+n} - n\, u_{x+n-1} + \frac{n\,(n-1)}{1.2} u_{x+n-2} - \text{\&c.}$$

COR. 1. If $n = -1$; $\Delta^{-1} u_x = u_{x-1} + u_{x-2} + u_{x-3} + \text{\&c.}$

or $\Sigma u_x = u_{x-1} + u_{x-2} + u_{x-3} + \text{\&c.}$ together with an arbitrary constant;

or $u_x = \Delta u_{x-1} + \Delta u_{x-2} + \Delta u_{x-3} + \text{\&c.}$

COR. 2. If $n = -2$; $\Sigma^2 u_x = u_{x-2} + 2\, u_{x-3} + 3\, u_{x-4} + \text{\&c.}$ together with $A + Bx$.

$$(2). \quad \Delta^n u_x = (-1)^n (1 - e^{\frac{d}{dx}})^n u_x = (-1)^n \left(1 - n\, e^{\frac{d}{dx}} + \frac{n\,(n-1)}{1.2} e^{2\frac{d}{dx}} - \text{\&c.}\right) u_x$$

$$= (-1)^n \left(u_x - n\, u_{x+1} + \frac{n\,(n-1)}{1.2} u_{x+2} - \text{\&c.}\right)$$

COR. 1. If $n = -1$, $\Delta^{-1} u_x = \Sigma u_x = -(u_x + u_{x+1} + u_{x+2} + \text{\&c.})$; to which we may add an arbitrary constant.

COR. 2. If $n = -2$, $\Sigma^2 u_x = u_x + 2\, u_{x+1} + 3\, u_{x+2} + \text{\&c.}$, together with $A + Bx$.

$$(3). \quad \Delta^n u_x = e^{\frac{n\,d}{dx}} \left(\frac{e^{\frac{d}{dx}} - 1}{e^{\frac{d}{dx}}}\right)^n u_x = e^{\frac{n\,d}{dx}} \left(\frac{e^{\frac{d}{dx}}}{e^{\frac{d}{dx}} - 1}\right)^{-n} u_x$$

$$= e^{n\frac{d}{dx}} \left(1 + (e^{\frac{d}{dx}} - 1)^{-1}\right)^{-n} u_x$$

$$=e^{n\frac{d}{dx}}\left\{1-n\left(e^{\frac{d}{dx}}-1\right)^{-1}+\frac{n(n+1)}{1.2}\left(e^{\frac{d}{dx}}-1\right)^{-2}-\&c.\right\}$$

$$=e^{n\frac{d}{dx}}\left\{u_x-n\,\Delta^{-1}\,u_x+\frac{n(n+1)}{1.2}\,\Delta^{-2}\,u_x-\&c.\right\}$$

$$=u_{x+n}-n\,\Delta^{-1}\,u_{x+n}+\frac{n(n+1)}{1.2}\,\Delta^{-2}\,u_{x+n}-\&c.$$

or $$=u_{x+n}-n\,\Sigma\,u_{x+n}+\frac{n(n+1)}{1.2}\,\Sigma^2\,u_{x+n}-\&c.$$

(4). $$\Delta^n\,u_x=(-1)^{-n}\left(1-\frac{e^{\frac{d}{dx}}}{e^{\frac{d}{dx}}-1}\right)^{-n}u_x$$

$$=(-1)^{-n}\left\{1+n\,e^{\frac{d}{dx}}\left(e^{\frac{d}{dx}}-1\right)^{-1}+\frac{n(n+1)}{1.2}\,e^{\frac{2d}{dx}}\left(e^{\frac{d}{dx}}-1\right)^{-2}+\&c.\right\}u_x$$

$$=(-1)^{-n}\left\{u_x+n\,\Sigma\,u_{x+1}+\frac{n(n+1)}{1.2}\,\Sigma^2\,u_{x+2}+\&c.\right\}$$

(5). $$\Delta^n\,u_x=\left(\frac{e^{\frac{d}{dx}}}{e^{\frac{d}{dx}}-1}-1\right)^{-n}u_x$$

$$=\left\{e^{-n\frac{d}{dx}}\left(e^{\frac{d}{dx}}-1\right)^n+n\,e^{-(n+1)\frac{d}{dx}}\left(e^{\frac{d}{dx}}-1\right)^{n+1}+\&c.\right\}u_x$$

$$=\Delta^n\,u_{x-n}+n\,\Delta^{n+1}\,u_{x-n-1}+\frac{n(n+1)}{1.2}\,\Delta^{n+2}\,u_{x-n-2}+\&c.$$

These formulæ are all quite independent of the value of n, and serve to connect the nth difference of a function of x with differences of functions of $x+n$, &c., $x+1$, &c.

We shall now obtain the converse series of connections, those of u_{x+n} with u_x, &c.

(6). $$u_{n+n}=(1+\Delta)^n\,u_x=\left(1+n\,\Delta+\frac{n(n-1)}{1.2}\,\Delta^2+\&c.\right)u_x$$

$$=n_x+n\,\Delta\,u_x+n\,\frac{(n-1)}{1.2}\,\Delta^2\,n_x+\&c.$$

(7). $$u_{x+n}=(\Delta+1)^n u_x=\left(\Delta^n+n\Delta^{n-1}+n\,\frac{(n-1)}{1.2}\,\Delta^{n-2}+\&c.\right)u_x$$

$$=\Delta^n\,u_x+n\,\Delta^{n-1}\,u_x+\frac{n(n-1)}{1.2}\,\Delta^{n-2}u_x+\&c.$$

If n were a positive integer, formula (1) would coincide with formula (2); and formula (6) with formula (7); but in our present calculus they are by no means the same thing.

(8.) $$u_{x+n}=\left(\frac{1}{1+\Delta}\right)^{-n}u_x=\left(1-\frac{\Delta}{1+\Delta}\right)^{-n}u_x$$

$$=\left(1+n\frac{\Delta}{1+\Delta}+\frac{n(n+1)}{1.2}\frac{\Delta^2}{(1+\Delta)^2}+\text{\&c.}\right)u_x$$

$$=u_x+n\,\Delta\,u_{x-1}+\frac{n(n+1)}{1.2}\Delta^2\,u_{x-2}+\text{\&c.}$$

Cor. If $n=1$, $u_{x+1}=u_x+\Delta\,u_{x-1}+\Delta^2\,u_{x-2}+\text{\&c.}$

(9.) $$u_{x+n}=(-1)^{-n}\left(\frac{\Delta}{1+\Delta}-1\right)^{-n}u_x$$

$$=(-1)^{-n}\left(\Delta^{-n}(1+\Delta)^n+n\,\Delta^{-(n+1)}(1+\Delta)^{n+1}+\text{\&c.}\right)u_x$$

$$=(-1)^{-n}\left\{\Sigma^n u_{x+n}+n\,\Sigma^{n+1}u_{x+n+1}+\frac{n(n+1)}{1.2}\Sigma^{n+2}u_{x+n+2}+\text{\&c.}\right\}$$

In strictness we ought to write Δ^{-n} for Σ^n, but the latter notation is more familiar to the eye.

(10.) $$u_{x+n}=\Delta^n\left(\frac{1+\Delta}{\Delta}\right)^n u_x=\Delta^n\left(1-\frac{1}{1+\Delta}\right)^{-n}u_x$$

$$=\Delta^n\left\{1+n\,(1+\Delta)^{-1}+\frac{n(n+1)}{1.2}(1+\Delta)^{-2}+\text{\&c.}\right\}u_x$$

$$=\Delta^n u_x+n\,\Delta^n\,u_{x-1}+\frac{n(n+1)}{1.2}\Delta^n\,u_{x-2}+\text{\&c.}$$

Formula (10) is a particular form of formula (1), for by formula (1), $\Delta^{-m}u_{x+m}=u_x+m\,u_{x-1}+\text{\&c.}$, which is reduced to (10) by multiplying by Δ^m. In the same manner we may reproduce formula (2.)

The last class of relations which we shall produce are such as do not depend on the general expansion of the binomial.

(11.) $$u_{x+n}=e^{n\frac{d}{dx}}u_x=e^{n\frac{d}{dx}}\,\frac{e^{\frac{d}{dx}}-1}{e^{\frac{d}{dx}}-1}\,u_x$$

$$=(e^{\frac{d}{dx}}-1)\,(e^{(n-1)\frac{d}{dx}}+e^{(n-2)\frac{d}{dx}}+e^{(n-3)\frac{d}{dx}}+\text{\&c.})\,u_x$$

$$=(e^{\frac{d}{dx}}-1)\,(u_{x+n-1}+u_{x+n-2}+u_{x+n-3}+\text{\&c.})$$

$$=\Delta\,u_{x+n-1}+\Delta\,u_{x+n-2}+\Delta\,u_{x+n-3}+\text{\&c.}$$

Cor. 1. If $n=0$; $u_x=\Delta\,u_{x-1}+\Delta\,u_{x-2}+\Delta\,u_{x-3}+\text{\&c.}$, which coincides with Cor. 1., formula (1.)

Cor. 2. If n be a positive integer

$$u_{x+n}=\Delta\,u_{x+n-1}+\Delta\,u_{x+n-2}+\text{\&c.}\;\;+\Delta\,u_{x+1}+\Delta\,u_x+\Delta\,u_{x-1}+\text{\&c.}$$

$$=\Delta\,u_{x+n-1}+\Delta\,u_{x+n-2}+\text{\&c.}\;\;+\Delta\,u_{x+1}+\Delta\,u_x+u_x;\text{ by Cor. 1.}$$

(12.)
$$u_{x+n} = -e^{n\frac{d}{dx}} \frac{e^{\frac{d}{dx}}-1}{1-e^{\frac{d}{dx}}} u_x$$

$$= -(e^{\frac{d}{dx}}-1)\,(e^{n\frac{d}{dx}}+e^{(n+1)\frac{d}{dx}}+e^{(n+2)\frac{d}{dx}}+\&c.)\,u_x$$

$$= -(\Delta u_{x+n}+\Delta u_{x+n+1}+\Delta u_{x+n+2}+\&c.)$$

Cor. If $n=0$, $\Sigma u_x = -(u_x+u_{x+1}+u_{x+2}+\&c.)$,

which coincides with Cor. 1, formula (2.)

(13.)
$$u_{x+n} = e^{(n-1)\frac{d}{dx}} e^{\frac{d}{dx}} u_x = e^{(n-1)\frac{d}{dx}} \frac{1}{1-\dfrac{e^{\frac{d}{dx}}-1}{e^{\frac{d}{dx}}}} u_x$$

$$= e^{(n-1)\frac{d}{dx}} \left(1+\frac{e^{\frac{d}{dx}}-1}{e^{\frac{d}{dx}}}+\left(\frac{e^{\frac{d}{dx}}-1}{e^{\frac{d}{dx}}}\right)^2+\&c.\right) u_x$$

$$= u_{x+n-1}+\Delta u_{x+n-2}+\Delta^2 u_{x+n-3}+\&c.$$

which coincides with the Cor. to formula (8).

Thus formulæ (1), (2), and (8), include formulæ (11), (12), and (13).

It is evident that by the same process all the ordinary formulæ in finite differences, which are usually obtained by the aid of generating functions, may be easily obtained.

For example the following:

(14.)
$$u_{x+n} = (n+1)\left\{u_x+\frac{(n+1)^2-1^2}{1.2.3}\Delta^2 u_{x-1}+\frac{\overline{(n+1)^2-1^2}\,.\,\overline{(n+1)^2-2^2}}{1.2.3.4.5}\Delta^4 u_{x-2}+\&c.\right\}$$

$$-n\left\{u_{x-1}+\frac{n^2-1^2}{1.2.3}\Delta^2 u_{x-2}+\&c.\right\}$$

We have
$$\frac{1}{1-a(1+\Delta)} = \frac{1-\dfrac{a}{1+\Delta}}{\left\{1-a\,(1+\Delta)\right\}\left\{1-\dfrac{a}{1+\Delta}\right\}} =$$

$$\frac{1}{(1-a)^2-\dfrac{a\,\Delta^2}{1+\Delta}} - \frac{\dfrac{a}{1+\Delta}}{(1-a)^2-\dfrac{a\,\Delta^2}{1+\Delta}}$$

Now $\dfrac{1}{(1-a)^2-az}$ when expanded in terms of a, gives as the coefficient of a^n,

$$(n+1)\left\{1+\frac{(n+1)^2-1^2}{1.2.3}z+\&c.\right\}$$

Hence, if we equate the coefficients of a^n in the two equivalent expressions

$$\frac{1}{1-a(1+\Delta)}u_x \text{ and } \left(\frac{1}{(1-a)^2-\frac{a\Delta^2}{1+\Delta}}-\frac{\frac{a}{1+\Delta}}{(1-a)^2-\frac{a\Delta^2}{1+\Delta}}\right)u_x$$

the result will be

$$(1+\Delta)^n u_x=(n+1)\left\{1+\frac{(n+1)^2-1^2}{1.2.3}\frac{\Delta^2}{1+\Delta}+\&c.\right\}u_x$$

$$-n\left\{1+\frac{n^2-1^2}{1.2.3}\frac{\Delta^2}{1+\Delta}+\&c.\right\}\frac{1}{1+\Delta}u_x$$

or

$$u_{x+n}=(n+1)\left\{u_x+\frac{(n+1)^2-1^2}{1.2.3}\Delta^2 u_{x-1}+\&c.\right\}$$

$$-n\left\{u_{x-1}+\frac{n^2-1^2}{1.2.3}\Delta^2 u_{x-2}+\&c.\right\}$$

24. Let us apply these formulæ to examples.

Ex. 1. Let $u_x=e^{ax}$, then

$$\Delta^n u_x=(e^{\frac{d}{dx}}-1)^n e^{ax}=(e^a-1)^n e^{ax} \text{ (by A.)}$$

Ex. 2. Let $u_x=x$, $n=\frac{1}{2}$, then, formula (2),

$$\Delta^{\frac{1}{2}}x=\sqrt{-1}\left\{x-\tfrac{1}{2}(x+1)+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}(x+2)-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}(x+3)+\&c.\right\}$$

$$=\sqrt{-1}\left\{x-\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}x-\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}x+\&c.\right.$$

$$-\tfrac{1}{2}\left(1+\frac{\frac{1}{2}}{1}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}+\&c.\right)$$

$$=\sqrt{-1}\,x(1-1)^{\frac{1}{2}}-\frac{\sqrt{-1}}{2}(1-1)^{-\frac{1}{2}}$$

$$=\infty$$

Ex. 3. $$\Delta^n x=(-1)^n\left(x-n(x+1)+\frac{n(n-1)}{1.2}(x+2)-\&c.\right)$$

$$=(-1^n x(1-1)^n-n(-1)^n\left(1-\frac{n-1}{1}+\frac{(n-1)(n-2)}{1.2}-\&c.\right)$$

$$=(-1)^n x(1-1)^n-n(-1)^n(1-1)^{n-1}$$

which is zero when n is greater than 1, finite only when $n=1$, in which case it is 1; and infinite when n is less than 1.

It is evident that this introduction of ∞ may indicate simply that the form of the expansion is incorrect: for $\Delta\,\infty\,x=\infty(x+1)-\infty\,x=x+\text{const.}$ is the analytical result of the equation $\Delta\,x=(x+1)-x=0\,x.+\text{const.}$, by dividing both sides by the symbol 0.

When n is less than 1, therefore, it is necessary to seek some other method of obtaining the nth difference. The following method, analogous to that by which we obtained the nth differential coefficient of a logarithm in Art. 2 appears to be the most simple.

Let x be represented by $\frac{e^{px}-e^{qx}}{p}$, where q is of a higher order than p, and both are 0.

Then
$$\Delta^n x = \frac{(e^p-1)^n e^{px}-(e^q-1)^n e^{qx}}{p}$$
$$=\frac{(p+\frac{p^2}{1.2}+\frac{p^3}{1.2.3}+\&c.)^n e^{px}-(q+\&c.)e^{qx}}{p}$$
$$=\frac{(p+\frac{p^2}{1.2}+\&c.)^n(1+px+\&c.)-(q+\frac{q^2}{1.2}+\&c.)^n(1+qx+\&c.)}{p}$$
$$=\frac{p^n+\frac{np^{n+1}}{2}+\frac{n(3n+1)}{24}p^{n+2}+\&c.-q^n-\frac{nq^{n+1}}{2}-\frac{n(3n+1)}{24}q^{n+2}+\&c.}{p}$$
$$+\frac{p+\frac{n+1\,n\,p^{n+2}}{2}+\&c.-q^{n+1}-\frac{nq^{n+2}}{2}-\&c.}{p}x$$
$$+\frac{\frac{p^{n+2}}{2}+\&c.-\frac{q^{n+2}}{2}+\&c.}{p}x^2$$
$$+\&c.$$

If $n>0<1$, every part vanishes except the constant, which is infinite: if $n=1$, $\Delta x=1$; if $n>1$, every term is zero.

If n is negative, there will still exist the infinite constant which may be regarded as part of the arbitrary constant; there will also exist in some instances infinite functions of x, which, as will easily be seen, may be considered in those cases as part of the arbitrary functions.

Let $n=-1$; then
$$\Delta^{-1}x = \text{const.}-\frac{\frac{1}{2}p+\&c.}{p}x+\tfrac{1}{2}x^2$$
$$=\text{const.}+\frac{x(x-1)}{2}$$

Let $n=-2$; and
$$\Delta^{-2}x=\text{const.}+\text{const.}\,x-\frac{x^2}{2}+\frac{x^3}{6}$$
and so on.

Ex. 4. $\Delta^n x^m=(x+n)^m-n(x+n-1)^m+\frac{n(n-1)}{1.2}(x+n-2)^m-\&c.$

by the first formula.

Ex. 5. $$\Delta^n \frac{1}{(a+x)^m} = (-1)^n \left\{ \frac{1}{(a+x)^m} - \frac{n}{(a+x+1)^m} + \&c. \right\}$$

$$= (-1)^n \Delta_{x'}{}^n \frac{1}{x'^m}; \text{ if } x' = a + x$$

Ex. 6. *To find* $\Delta^{\frac{1}{2}} \frac{1}{x}$.

By formula (2), $\Delta^{\frac{1}{2}} \frac{1}{x} = \sqrt{-1} \left\{ \frac{1}{x} - \frac{\frac{1}{2}}{x+1} - \frac{1 \,.\, 1}{2 \,.\, 4} \frac{1}{x+2} - \&c. \right\}$

Let $$v = \frac{y^x}{x} - \frac{1}{2} \frac{y^{x+1}}{x+1} - \&c.$$

then $$\frac{dv}{dy} = y^{x-1} \left(1 - \frac{1}{2} y - \frac{1 \,.\, 1}{2 \,.\, 4} \,.\, y^2 \;\&c.\right)$$

$$= y^{x-1} (1-y)^{\frac{1}{2}}$$

and $$\Delta^{\frac{1}{2}} \frac{1}{x} = \sqrt{-1} \int_0^1 y^{x-1} (1-y)^{\frac{1}{2}} dy$$

$$= \sqrt{-1}\, \Gamma(x, \tfrac{3}{2}) = \sqrt{-1} \frac{\Gamma x \, \Gamma \frac{3}{2}}{\Gamma(x+\frac{3}{2})}$$

Ex. 7. *To find* $\Delta^n \frac{1}{x}$.

$$\Delta^n \frac{1}{x} = (-1)^n \left\{ \frac{1}{x} - \frac{n}{x+1} + \frac{n(n-1)}{1 \,.\, 2} \frac{1}{x+2} - \&c. \right\}$$

Let $$v = \frac{y^x}{x} - \frac{n\, y^{x+1}}{x+1} + \&c.$$

then $$\frac{dv}{dy} = y^{x-1} (1-y)^n$$

and $$\Delta^n \frac{1}{x} = (-1)^n \int_0^1 y^{x-1} (1-y)^n dy$$

$$= (-1)^n\, \Gamma(x, n+1) = (-1)^n \frac{\Gamma x \, \Gamma(n+1)}{\Gamma(x+n+1)}$$

Cor. 1. If n be a whole number, $\Gamma(x+n+1) = x(x+1) \ldots (x+n)\, \Gamma x$

$$\therefore \qquad \Delta^n \frac{1}{x} = (-1)^n \frac{1 \,.\, 2 \cdots n}{x(x+1) \ldots (x+n)}$$

Cor. 2. If $n = \frac{1}{2}$, $\Delta^{\frac{1}{2}} \frac{1}{x} = \sqrt{-1} \frac{\Gamma x \, \Gamma \frac{3}{2}}{\Gamma(x-\frac{3}{2})} = \frac{1}{2} \sqrt{-1} \sqrt{\pi} \frac{\Gamma x}{\Gamma(x+\frac{3}{2})}$

Ex. 8. *To find*, $\Delta^n \frac{1}{x^m}$, *m being any integer*.

$$\Delta^n \frac{1}{x^m} = (-1)^n \left\{ \frac{1}{x^m} - n \frac{1}{(x+1)^m} + \frac{n(n-1)}{1 \,.\, 2} \frac{1}{(x+2)^m} - \&c. \right\} \text{ by formula (2).}$$

Let $$v=\frac{y^x}{x^m}-n\frac{y^{x+1}}{(x+1)^m}+\&c.$$

then $$\left(y\frac{d}{dy}\right)^m v=y^x(1-y)^n$$

$\therefore$ $$v=\int_0\frac{dy}{y}\int\frac{dy}{y}\ \ldots.\ (m \text{ times})\ldots y^x(1-y)^n.$$

and $$\Delta^n\frac{1}{x^m}=(-1)^n\int_0^1\frac{dy}{y}\int_0\frac{dy}{y}\int_0\frac{dy}{y}\ldots y^x(1-y)^n.$$

Ex. 9. *To find* $\Delta^n \sin a x$.

$$\Delta^n\sin a x=\frac{1}{2\sqrt{-1}}(e^{\frac{d}{dx}}-1)^n(e^{ax\sqrt{-1}}-e^{-ax\sqrt{-1}})$$

$$=\frac{1}{2\sqrt{-1}}\left\{(e^{a\sqrt{-1}}-1)^n e^{ax\sqrt{-1}}-(e^{-a\sqrt{-1}}-1)^n e^{-ax\sqrt{-1}}\right\}$$

$$=\frac{1}{2\sqrt{-1}}\left\{(e^{\frac{a}{2}\sqrt{-1}}-e^{-\frac{a}{2}\sqrt{-1}})^n e^{\frac{na}{2}\sqrt{-1}+ax\sqrt{-1}}\right.$$

$$\left.-(-1)^n(e^{\frac{a}{2}\sqrt{-1}}-e^{-\frac{a}{2}\sqrt{-1}})^n e^{-\frac{na}{2}\sqrt{-1}+ax\sqrt{-1}}\right\}$$

$$=\frac{1}{2\sqrt{-1}}\left(2\sqrt{-1}\sin\frac{a}{2}\right)^n\left\{\left(\cos\overline{ax+\frac{na}{2}}+\sqrt{-1}\sin\overline{ax+\frac{na}{2}}\right)\right.$$

$$\left.-(\cos\overline{2\lambda+1}n\pi-\sqrt{-1}\sin\overline{2\lambda+1}n\pi)\left(\cos\overline{ax+\frac{na}{2}}-\sqrt{-1}\sin\overline{ax+\frac{na}{2}}\right)\right\}$$

$$=(2\sqrt{-1})^{n-1}\sin^n\frac{a}{2}\left\{\cos\left(ax+\frac{na}{2}\right)+\sqrt{-1}\sin\left(ax+\frac{na}{2}\right)\right.$$

$$\left.-\cos\left(\overline{2\lambda+1}n\pi+ax+\frac{na}{2}\right)+\sqrt{-1}\sin\left(\overline{2\lambda+1}n\pi+ax+\frac{na}{2}\right)\right\}$$

$$=2^n(\sqrt{-1})^{n-1}\sin^n\frac{a}{2}\sin\left(ax+\frac{na}{2}+\frac{2\lambda+1}{2}n\pi\right)(\sin\overline{2\lambda+1}\frac{n\pi}{2}\ \&c.$$

$$=2^n(\cos nr\pi-\sqrt{-1}\sin nr\pi)\sin^n\frac{a}{2}\sin\left(ax+\frac{na}{2}+\frac{(2\lambda+1)}{2}n\pi\right);$$

r and λ being any integers.

Ex. 10. *To find* $\Delta^n e^{mx}\sin a x$.

$$\Delta^n e^{mx}\sin a x=\frac{1}{2\sqrt{-1}}(e^{\frac{d}{dx}}-1)^n(e^{mx+ax\sqrt{-1}}-e^{mx-ax\sqrt{-1}})$$

$$=\frac{1}{2\sqrt{-1}}\{(e^{m+a\sqrt{-1}}-1)^n e^{mx+ax\sqrt{-1}}-(e^{m-a\sqrt{-1}}-1)^n e^{mx-ax\sqrt{-1}}\}$$

$$=\frac{e^{mx}}{2\sqrt{-1}}\{(e^m\cos a+e^m\sqrt{-1}\sin a-1)^n(\cos a x+\sqrt{-1}\sin a x)$$

$$-(e^m \cos a - e^m \sqrt{-1} \sin a - 1)^n (\cos a x - \sqrt{-1} \sin a x)\}$$

Let $\qquad e^m \cos a - 1 = \text{P} \cos \theta,\ e^m \sin a = \text{P} \sin \theta;$

then $\qquad \text{P}^2 = e^{2m} - 2 e^m \cos a + 1$, and $\tan \theta = \dfrac{e^m \sin a}{e^m \cos a - 1}$

$$\therefore\ \Delta^n e^{mx} \sin a x = \frac{e^{mx}}{2\sqrt{-1}} \text{P}^n \{(\cos n \overline{2\lambda\pi + \theta} + \sqrt{-1} \sin n \overline{2\lambda\pi + \theta}) (\cos a x + \sqrt{-1} \sin a x)$$

$$-(\cos \overline{2\lambda'\pi + \theta} - \sqrt{-1} \sin \overline{2\lambda'\pi + \theta}) (\cos a x - \sqrt{-1} \sin a x)\}$$

$$= \frac{e^{mx} \text{P}^n}{\sqrt{-1}} \{\sqrt{-1} \cos n (\lambda' - \lambda) \pi + \sin n (\lambda' - \lambda) \pi\} \sin (a x + n \theta + n \overline{\lambda + \lambda'} \pi)$$

$$= e^{mx} \text{P}^n (\cos r n \pi - \sqrt{-1} \sin r n \pi) \sin (a x + n \theta + n \lambda \pi)$$

r and λ being any integers.

Similar expressions may be obtained for the nth differences of $\cos a x$ and of $e^{mx} \cos a x$.

25. We shall now proceed to the demonstration of certain theorems analogous to those in the ordinary calculus of differences.

PROP. 1. $\qquad \dfrac{1}{v_{x+n}} = \dfrac{1}{v_x} - \dfrac{n b}{v_x v_{x+1}} + \dfrac{n(n-1) \cdot b^2}{v_x v_{x+1} v_{x+2}} - \text{\&c.}$

where $v_x = a + b x$ or $\Delta v_x = b$.

By formula (6); putting $\dfrac{1}{v_x}$ for u_x

$$\frac{1}{v_{x+n}} = \frac{1}{v_x} + n \Delta \frac{1}{v_x} + \frac{n(n-1)}{1 \cdot 2} \Delta^2 \frac{1}{v_x} + \text{\&c.}$$

Now $\qquad \Delta \dfrac{1}{v_x} = -\dfrac{\Delta v_x}{v_x v_{x+1}} = -\dfrac{b}{v_x v_{x+1}},\ \Delta^2 \dfrac{1}{v_x} = \dfrac{1 \cdot 2 \cdot b^2}{v_x v_{x+1} v_{x+2}}$ &c, &c.

$$\therefore \qquad \frac{1}{v_{x+n}} = \frac{1}{v_x} - \frac{n b}{v_x v_{x+1}} + \frac{n(n-1) b^2}{v_x v_{x+1} v_{x+2}} \text{ \&c.}$$

PROP. 2. A similar result may be obtained from formula (8).

For $\qquad \Delta \dfrac{1}{v_{x-1}} = -\dfrac{b}{v_x v_{x-1}},\ \Delta^2 \dfrac{1}{v_{x-2}} = \dfrac{1 \cdot 2 \cdot b^2}{v_x v_{x-1} v_{x-2}}$ &c.

$$\therefore \qquad \frac{1}{v_{x+n}} = \frac{1}{v_x} - \frac{n b}{v_x v_{x-1}} + \frac{n(n+1) b^2}{v_x v_{x-1} v_{x-2}} - \text{\&c.}$$

PROP. 3. $\qquad \Delta^n u_x v_x = v_x \Delta^n u_x + n \Delta v_x \Delta^{n-1} u_{x+1} + \text{\&c.}$

For $\{(1+\Delta)(1+\Delta') - 1\} u_x v_x$ being an operation on $u_x v_x$ may be repeated according to any law, consequently

$\Delta^n u_x v_x = \{(1+\Delta)(1+\Delta') - 1\}^n u_x v_x$: and every step in the demonstration is the same as when n is a whole number.

COR. The same is true of the formula for the nth difference of $u_{x,y}$: for

$$\Delta u_{x,y} = \{(1+\Delta_x)(1+\Delta_y)-1\}u_{x,y}$$

$$\therefore \quad \Delta^n u_{x,y} = \{\overline{1+\Delta_x}\,\overline{1+\Delta_y}-1\}^n u_{x,y}$$

or $$= (e^{\frac{d}{dx}+\frac{d}{dy}}-1)^n u_{x,y}$$

and the same results are produced as when n is a whole number.

PROP. 4. $$\mathrm{F}(\Delta)\, e^{rx} f(x) = e^{rx}\,\mathrm{F}(e^r\,\overline{1+\Delta}-1) f(x)$$

Let $u_x = e^{rx}$; and $v_x = f(x)$ in Prop. 3.

$$\therefore \quad \Delta^n e^{rx} f(x) = f(x)\,\Delta^n e^{rx} + n\,\Delta f x\,\Delta^{n-1} e^{rx+r} + \&c.$$

$$= f(x)(e^r-1)^n e^{rx} + n\,\Delta f x\,(e^r-1)^{n-1} e^{rx+r} + \&c.$$

$$= e^{rx}(e^r-1+e^r\,\Delta)^n f(x)$$

$$= e^{rx}(e^r\overline{1+\Delta}-1)^n f(x)$$

which being true for all values of n, shews that the following theorem is also true:

$$\mathrm{F}(\Delta)\,.\,e^{rx} f(x) = e^{rx}\,\mathrm{F}(e^r\,\overline{1+\Delta}-1)\,.\,f(x)$$

PROP. 5. $$\Delta u_x = \left(\left(1+\frac{1}{x}\right)^{\mathrm{D}}-1\right)u_\theta = \left\{(1+\Delta_\theta)^{\log\left(1+\frac{1}{x}\right)}-1\right\}u_\theta$$

Let $x=e^\theta$, and let u_x be represented by u_θ when e^θ is written for x, let also Δ_θ be the symbol of difference $u_{\theta+1}-u_\theta$. Then by (C), when n is an integer,

$$x^n\frac{d^n u_x}{dx^n} = \mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)\,\ldots\,(\mathrm{D}-n+1)\,u_\theta$$

$$\therefore \quad \Delta u_x = (e^{\frac{d}{dx}}-1)\,u_x = \left(\frac{d}{dx}+\frac{1}{1\,.\,2}\left(\frac{d}{dx}\right)^2+\frac{1}{1\,.\,2\,.\,3}\left(\frac{d}{dx}\right)^3+\&c.\right)u_x$$

$$= \left(\frac{1}{x}\mathrm{D}+\frac{1}{1\,.\,2}\frac{1}{x^2}\mathrm{D}(\mathrm{D}-1)+\frac{1}{1\,.\,2\,.\,3}\frac{1}{x^3}\mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)+\&c.\right)u_\theta$$

$$= \left(\left(1+\frac{1}{x}\right)^{\mathrm{D}}-1\right)u_\theta$$

But $$\Delta_\theta\,u = (e^{\mathrm{D}}-1)\,u_\theta \qquad \therefore \quad e^{\mathrm{D}}u_\theta = (1+\Delta_\theta)\,u_\theta$$

or $$\mathrm{D} = \log(1+\Delta_\theta).$$

Hence $$\left(1+\frac{1}{x}\right)^{\mathrm{D}} = \left(1+\frac{1}{x}\right)^{\log(1+\Delta_\theta)} = (1+\Delta_\theta)^{\log\left(1+\frac{1}{x}\right)}$$

and $$\Delta u_x = \left(\left(1+\frac{1}{x}\right)^{\mathrm{D}}-1\right)u_\theta = \left\{(1+\Delta_\theta)^{\log\left(1+\frac{1}{x}\right)}-1\right\}u_\theta$$

COR. $$u_{x+n} = (1+\Delta_\theta)^{\log\left(1+\frac{1}{x}\right)}u_\theta$$

PROP. 6. $$u_{x+n}=(1+\Delta_\theta)^{\log\left(1+\frac{n}{x}\right)}u_\theta$$

For $$u_{x+n}=e^{n\frac{d}{dx}}u_x=(1+n\frac{d}{dx}+\frac{n^2\left(\frac{d}{dx}\right)^2}{1.2}+\&c.)\,u_\theta$$

$$=(1+\frac{n}{x}\,\mathrm{D}+\frac{1}{1\,.\,2}\,\frac{n^2}{x^2}\,\mathrm{D}\,(\mathrm{D}-1)+\&c.)u_\theta$$

$$=\left(\left(1+\frac{n}{x}\right)^{\mathrm{D}}-1\right)u_\theta$$

$$=(1+\Delta_\theta)^{\log\left(1+\frac{n}{x}\right)}u_\theta$$

It must be observed that x is considered constant with respect to Δ_θ in the formula $\left(1+\frac{1}{x}\right)^{\log(1+\Delta_\theta)}$. Had we supposed it otherwise, we must have taken account of the differential coefficients of $\frac{1}{x}$ itself. This would have given the following theorem.

PROP. 7. $$\Delta\,u_x=\{(1-e^{-\theta})^{-(\mathrm{D}+1)}-1\}u_\theta$$

$$=\{(1+\Delta_\theta)^{-\log(1-e^{-\theta})}(1-e^{-\theta})^{-1}-1\}u_\theta$$

For $$\Delta\,u_x=(e^{\frac{d}{dx}}-1)\,u_x=\left(\frac{d}{dx}+\frac{1}{1\,.\,2}\left(\frac{d}{dx}\right)^2+\&c.\right)u_x$$

$$=\left(\frac{1}{x}\,\mathrm{D}+\frac{1}{1\,.\,2}\,\frac{1}{x^2}\,\mathrm{D}\,(\mathrm{D}-1)+\&c.\right)u_\theta$$

$$=\left((\mathrm{D}+1)e^{-\theta}+\frac{1}{1\,.\,2}(\mathrm{D}+2)\,(\mathrm{D}+1)\,e^{-2\theta}+\&c.\right)u_\theta \qquad \text{(B)}$$

$$=\{(1-e^{-\theta})^{-(\mathrm{D}+1)}-1\}u_\theta$$

$$=\{(1-e^{-\theta})^{-\log(1+\Delta_\theta)-1}-1\}u_\theta$$

$$=\{(1+\Delta_\theta)^{-\log(1-e^{-\theta})}(1-e^{-\theta})^{-1}-1\}u_\theta$$

PROP. 8. $$u_{x+n}=e^{n\frac{d}{dx}}u_x=\left(1+n\,\frac{d}{dx}+\frac{n^2}{1.2}\left(\frac{d}{dx}\right)^2+\&c.\right)u_x$$

$$=\left(1+n\,(\mathrm{D}+1)\,e^{-\theta}+\frac{n^2}{1\,.\,2}(\mathrm{D}+2)\,(\mathrm{D}+1)e^{-2\theta}+\&c.\right)u_\theta$$

$$=(1-n\,e^{-\theta})^{-(\mathrm{D}+1)}\,u_\theta$$

$$=(1+\Delta_\theta)^{-\log(1-n\,e^{-\theta})}\,(1-n\,e^{-\theta})^{-1}u_\theta$$

It is manifest that these formulæ do not follow the distributive law. They cannot, consequently, be applied with any great advantage to the solution of equations of differences. We shall exhibit their application only in one instance.

Equations of Differences.

26. As the method of solving equations of differences of the second and higher orders, by treating the symbols of operation as symbols of quantity, and reducing the resulting fraction by decomposing it into partial fractions, has been little, if at all, employed, we shall commence with an example or two in ordinary equations of differences.

Ex. 1. $$u_{x+3}+a\,u_{x+2}+b\,u_{x+1}+c\,u_x=\mathrm{X}.$$

This may be written

$$\{(1+\Delta)^3+a(1+\Delta)^2+b(1+\Delta)+c\}u_x=\mathrm{X}.$$

If we write $\Delta_{,}$ for $1+\Delta$, and suppose α, β, γ the roots of the equation $\Delta_{,}^3+a\,\Delta_{,}^2+b\,\Delta_{,}+c=0$; we get $(\Delta_{,}-\alpha)(\Delta_{,}-\beta)(\Delta_{,}-\gamma)u_x=\mathrm{X}$, or

$$u_x=\frac{1}{(\Delta_{,}-\alpha)(\Delta_{,}-\beta)(\Delta_{,}-\gamma)}\cdot(\mathrm{X}+0).$$

This equation is reduced, by the decomposition of the fraction of operation into its equivalent partial fractions, to

$$u_x=\frac{1}{(\alpha-\beta)(\alpha-\gamma)}\frac{1}{(\Delta_{,}-\alpha)}(\mathrm{X}+0)+\frac{1}{(\beta-\alpha)(\beta-\gamma)}\frac{1}{(\Delta_{,}-\beta)}(\mathrm{X}+0)$$
$$+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{(\Delta_{,}-\gamma)}(\mathrm{X}+0).$$

Now $\frac{1}{\Delta_{,}-\alpha}(\mathrm{X}+0)$ is the solution of the equation $v_{x+1}-\alpha v_x=\mathrm{X}+0$;

hence it is equal to $\alpha^x\left(\mathrm{A}+\Sigma\frac{\mathrm{X}}{\alpha^{x+1}}\right)$; and similarly of the others.

Hence the complete solution of the given equation is

$$u_x=\frac{1}{(\alpha-\beta)(\alpha-\gamma)}\alpha^x\left(\mathrm{A}+\Sigma\frac{\mathrm{X}}{\alpha^{x+1}}\right)+\frac{1}{(\beta-\alpha)(\beta-\gamma)}\beta^x\left(\mathrm{B}+\Sigma\frac{\mathrm{X}''}{\beta^{x+1}}\right)$$
$$+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}\gamma^x\left(\mathrm{C}+\Sigma\frac{\mathrm{X}}{\gamma^{x+1}}\right)$$

Cor. If $\alpha=\beta$, we must, as in similar cases, put $\alpha+c$ for β, and expand in terms of c. The result is

$$u_x=-\frac{1}{c(\alpha-\gamma)}\alpha^x\left(\mathrm{A}+\Sigma\frac{\mathrm{X}}{\alpha^{x+1}}\right)+$$
$$\frac{1}{\alpha-\gamma}\left(\frac{\alpha^x}{c}+x\,\alpha^{x-1}-\frac{\alpha^x}{\alpha-\gamma}\right)\left(\mathrm{B}+\Sigma\frac{\mathrm{X}}{\alpha^{x+1}}-c\,\Sigma\frac{\mathrm{X}(x+1)}{\alpha^{x+2}}\right)$$
$$+\frac{1}{(\gamma-\alpha)^2}\gamma^x\left(\mathrm{C}+\Sigma\frac{\mathrm{X}}{\gamma^{x+1}}\right)$$

$$=\frac{A_{\prime}a^x}{a-\gamma}+\frac{B\,x\,a^{x-1}}{a-\gamma}+\left(\frac{x\,a^{x-1}}{a-\gamma}-\frac{a^x}{(a-\gamma)^2}\right)\Sigma\frac{X}{a^{x+1}}-\frac{a^x}{a-\gamma}\Sigma\frac{X(x+1)}{a^{x+2}}$$

$$-\frac{1}{(\gamma-a)^2}\gamma^x\left(C+\Sigma\frac{X}{\gamma_{x+1}}\right).$$

In precisely the same manner we may integrate the general equation with constant coefficients.

Let us apply the formula of Prop. 6 to the following example.

Ex. 2. $\qquad u_x-3(x+1)\,u_{x+1}+2(x+1)(x+2)\,u_{x+2}=0.$

Calling $\Big/\left(1+\frac{1}{x}\right)\ l$ we get

$$u_\theta-3(e^\theta+1)(1+\Delta_\theta)^l u_\theta+2(e^\theta+1)(e^\theta+2)(1+\Delta_\theta)^{/\left(1+\frac{2}{x}\right)}u_\theta=0$$

Now since $\qquad e^{r\theta}(1+\Delta_\theta)^l u_\theta=e^{-rl}(1+\Delta_\theta)^l e^{r\theta}u_\theta$

we have

$$u_\theta-3(1+e^{-\theta})e^{-l}(1+\Delta_\theta)^l e^\theta u_\theta+2(e^\theta+1)(1+2e^{-\theta})e^{-/\left(1+\frac{2}{x}\right)}(1+\Delta_\theta)^{/\left(1+\frac{2}{x}\right)}e^\theta u_\theta=0.$$

or $\qquad u_\theta-3(1+\Delta_\theta)^l e^\theta u_\theta+2(e^\theta+1)(1+\Delta_\theta)^{/\left(1+\frac{2}{x}\right)}e^\theta u_\theta=0.$

Put $\qquad (1+\Delta_\theta)^l\,.\,(1+\Delta_\theta)^l\;$ for $\;(1+\Delta_\theta)^{/\left(1+\frac{2}{x}\right)}$

where Δ_θ in the former operates on the x in the latter.

$$\therefore\qquad u_\theta-3(1+\Delta_\theta)^l e^\theta u_\theta+2(1+e^{-\theta})e^\theta(1+\Delta_\theta)^l\,.\,(1+\Delta_\theta)^l e^\theta u_\theta=0$$

or $\qquad u_\theta-3(1+\Delta_\theta)^l e^\theta u_\theta+2(1+\Delta_\theta)^l e^\theta\,.\,(1+\Delta_\theta)^l e^\theta u_\theta=0$

or $\qquad u_\theta-3(1+\Delta_\theta)^l e^\theta u_\theta+2\,\overline{(1+\Delta_\theta)^l e^\theta}|^2 u_\theta=0$

which can be resolved into the two

$$\{1-(1+\Delta_\theta)^l e^\theta\}u_\theta=0 \text{ and } \{1-2(1+\Delta_\theta)^l e^\theta\}u_\theta=0$$

or $\qquad u_x-(x+1)\,u_{x+1}=0$ and $u_x-2(x+1)\,u_{x+1}=0$

where $\qquad u_x=\dfrac{A}{|\underline{x+1}}\;$ or $\;u_x=\dfrac{B}{2^x\,|\underline{x+1}}$

and therefore generally, $u_x=\dfrac{A+B\,2^{-x}}{|\underline{x+1}}$, which is the complete solution of the equation.

It is evident that the process employed in Example 1, applies equally in this Example, when a function of x appears on the right-hand side of the equation. Hence

Ex. 3. $\qquad u_x-3(x+1)\,u_{x+1}+2(x+1)(x+2)\,u_{x+2}=X$ gives

$$u_t = -\frac{X+0}{1-(1+\Delta_t)^t e^t} + 2\frac{X+0}{1-2(1+\Delta_t)^t e^t}$$

$$= \frac{A+B.2^{-x}}{\lfloor x+1} + \frac{1}{\lfloor x+1}\Sigma \lfloor x+1\, X - \frac{2^{-x+1}}{\lfloor x+1}\Sigma 2^x \lfloor x+1\, X$$

Cor. In the same manner may the more general equation

$u_x - a(x+1)^r u_{x+1} + b(x+1)^r (x+2)^r u_{x+2} + \&c. = X$, be solved.

It is not necesary to solve such equations as $u_{x+\frac{1}{2}} + a u_x + \&c. = X$, since it is evident that, by putting $x = \frac{1}{2} x'$, this form of equation is reduced to $v_{x'} + a v_{x'+1} + \&c. = X$, which has been already solved.

We proceed then to the solution of equations involving fractional differences. Here we must, at present, confine ourselves to very simple examples.

Ex. 4. $$\Delta^{\frac{1}{2}} u_x - a u_x = 0.$$

Since $$\Delta^{\frac{1}{2}} e^{m x} = (e^m - 1)^{\frac{1}{2}} e^{m x}$$

It is evident that if $m = \log (a^2+1)$ the solution of the equation is $u_x = A e^{m x}$

$$= A e^{x \log (a^2+1)} = A (a^2+1)^x$$

Ex. 5. $$\Delta^{\frac{1}{2}} u_x - a u_x = c e^{\alpha x}$$

$$u_x = A (a^2+1)^x + \frac{c}{(e^\alpha - 1)^{\frac{1}{2}} - a} e^{\alpha x}$$

or, if $$e^\alpha = 1 + b^2,\ u_x = A (a^2+1)^x + \frac{c}{b-a}(b^2+1)^x$$

Cor. If $b = a$, this solution fails. Put $b = a + \beta$ as in similar cases in Differential Equations:

then $$\frac{c}{b-a}(b^2+1)^x = \frac{c}{\beta}(a^2+1+2a\beta)^x = \frac{c}{\beta}(a^2+1)^x + \frac{c}{\beta} 2 a \beta x (a^2+1)^{x-1}$$

$$u_x = A_1 (a^2+1)^x + 2 a c x (a^2+1)^{x-1}$$

A_1 being an arbitrary constant.

It may be interesting to verify this solution.

$$\Delta^{\frac{1}{2}} (a^2+1)^x - a . (a^2+1)^x = 0 \text{ evidently,}$$

and $$\Delta^{\frac{1}{2}} 2 a c x (a^2+1)^{x-1} = 2 a c x\, a (a^2+1)^{x-1} + \tfrac{1}{2} 2 a c (a^2+1)^x \frac{1}{a} \text{ by Prop. 3.}$$

$$= 2 a^2 c x (a^2+1)^{x-1} + c (a^2+1)^x$$

$$\therefore \quad \Delta^{\frac{1}{2}} . 2 a c x (a^2+1)^{x-1} - a . 2 a c x (a^2+1)^{x-1} = c (a^2+1)^x = c e^{\alpha x} \text{ as it ought.}$$

Ex. 6. Generally, let $\Delta^{\frac{1}{2}} u_x - a u_x = X$

then $$u_x = A (a^2+1)^x + (\Delta^{\frac{1}{2}} - a)^{-1} X$$

$$=\mathrm{A}\,(a^2+1)^x+\frac{\Delta^{\frac{1}{2}}+a}{\Delta-a^2}\,\mathrm{X}$$

$$=\mathrm{A}\,(a^2+1)^x+(\Delta^{\frac{1}{2}}+a)\,(a^2+1)^x\,\Sigma\,\frac{\mathrm{X}}{(a^2+1)^{x+1}}$$

COR. 1. Let $\mathrm{X}=b\,x$; then the solution of the equation $\Delta^{\frac{1}{2}}u_x-a\,u_x=b\,x$ is

$$u_x=\mathrm{A}\,(a^2+1)^x-(\Delta^{\frac{1}{2}}+a)\,.\left(\frac{b\,x}{a^2}+\frac{b}{a^4}\right)$$

COR. 2. Let $\quad \mathrm{X}=b\,\dfrac{a^2\,x+a^2+1}{x\,(x+1)}$

then the value of $\quad (\Delta-a^2)^{-1}\,\mathrm{X}$ is $\;-\dfrac{b}{x}$

therefore the solution of the equation $\Delta^{\frac{1}{2}}u_x-a\,u_x=b\dfrac{a^2\,x+a^2+1}{x\,(x+1)}$

is
$$u_x=\mathrm{A}\,(a^2+1)^x-(\Delta^{\frac{1}{2}}+a)\,\frac{b}{x}$$

$$=\mathrm{A}\,(a^2+1)^x-\frac{a\,b}{x}-b\,\sqrt{-1}\,\frac{\overline{|x}\;\overline{|\tfrac{3}{2}}}{\overline{|x+\tfrac{3}{2}}}\quad\text{(by Ex. 6, Art. 24.)}$$

Ex. 7. $\qquad \Delta u_x+a\,\Delta^{\frac{1}{2}}u_x+b\,u_x=0.$

This equation may be written

$$(\Delta+a\,\Delta^{\frac{1}{2}}+b)\,u_x=0.$$

Let α, β be the roots of the equation $z^2+a\,z+b=0$, then

$$(\Delta^{\frac{1}{2}}-\alpha)\,(\Delta^{\frac{1}{2}}-\beta)\,u_x=0$$

$$u_x=\mathrm{A}\,(\Delta^{\frac{1}{2}}-\alpha)^{-1}\,.\,0+\mathrm{B}\,(\Delta^{\frac{1}{2}}-\beta)^{-1}\,.\,0$$

$$=\mathrm{A}\,(1+\alpha^2)^x+\mathrm{B}\,(1+\beta^2)^x\qquad\text{(Ex. 4.)}$$

COR. If $\alpha=\beta$, we obtain, by the usual process, $u_x=(\mathrm{A}+\mathrm{B}\,x)\,(1+\alpha^2)^x$.

Ex. 8. $\qquad \Delta\,u_x+a\,\Delta^{\frac{1}{2}}u_x+b\,u_x=c\,(1+e^2)^x$

The solution is

$$u_x=\frac{1}{\alpha-\beta}(\Delta^{\frac{1}{2}}-\alpha)^{-1}\,(0+c\,\overline{1+e^2}|^x)-\frac{1}{\alpha-\beta}(\Delta^{\frac{1}{2}}-\beta)^{-1}\,(0+c\,\overline{1+e^2}|^x)$$

$$=\mathrm{A}\,(1+\alpha^2)^x+\mathrm{B}\,(1+\beta^2)^x+\frac{c\,(1+e^2)^x}{(\alpha-\beta)\;(e-\alpha)}-\frac{c\,(1+e^2)^x}{(\alpha-\beta)\;(e-\beta)}$$

$$=\mathrm{A}\,(1+\alpha^2)^x+\mathrm{B}\,(1+\beta^2)^x+\frac{c\,(1+e^2)^x}{e^2+a\,e+b}$$

COR. If $e=\alpha$, we must proceed as in Cor. 3, Ex. 4, Class 1, of Differential Equations, and we shall obtain

$$u_x=\mathrm{A}\,(1+\alpha^2)^x+\mathrm{B}\,(1+\beta^2)^x+\frac{c\,x\,(1+\alpha^2)^{x-1}}{2\,\alpha+a}$$

Ex. 9. $$\Delta u_x + a\Delta^{\frac{1}{2}} u_x + b u_x = X$$

$$u_x = \frac{1}{a-\beta}(\Delta^{\frac{1}{2}} - a)^{-1}(0+X) - \frac{1}{a-\beta}(\Delta^{\frac{1}{2}} - \beta)^{-1}(0+X)$$

$$= A(1+a^2)^x + B(1+\beta^2)^x + \frac{1}{a-\beta}(\Delta^{\frac{1}{2}} - a)^{-1}.X - \frac{1}{a-\beta}(\Delta^{\frac{1}{2}} - \beta)^{-1}X$$

$$= A(1+a^2)^x + B(1+\beta^2)^x + \frac{1}{a-\beta}(\Delta^{\frac{1}{2}} + a)(1+a^2)^x \Sigma \frac{X}{(1+a^2)^{x+1}}$$

$$- \frac{1}{a-\beta}(\Delta^{\frac{1}{2}} + \beta)(1+\beta^2)^x \Sigma \frac{X}{(1+\beta^2)^{x+1}}$$

Ex. 10. $$u_x + a x \Delta^{\frac{1}{2}} u_x = X.$$

This equation gives $u_x = \dfrac{1}{1+a x \Delta^{\frac{1}{2}}} X$

$$= \frac{1 - a x \Delta^{\frac{1}{2}}}{1 - a^2 x \Delta^{\frac{1}{2}} x \Delta^{\frac{1}{2}}} X$$

$=(1 - a x \Delta^{\frac{1}{2}}) v_x$, where v_x is determined by the equation

$$v_x - a^2 x \Delta^{\frac{1}{2}} x \Delta^{\frac{1}{2}} v_x = X$$

Now $$\Delta^{\frac{1}{2}} x \Delta^{\frac{1}{2}} v_x = x\Delta v_x + \tfrac{1}{2} v_{x+1} \qquad \text{(Prop. 3.)}$$

$$\therefore \qquad v_x - a^2 x^2 \Delta v_x - \frac{a^2}{2} x v_{x+1} = X$$

or $$v_{x+1} - \frac{2}{a^2}\,\frac{1+a^2 x^2}{x+2x^2} v_x = -\frac{2X}{a^2(x+2x^2)}$$

which being solved by the ordinary method, v_x and therefore u_x (provided $\Delta^{\frac{1}{2}} v_x$ can be found) is known.

Equations of Differences with two independent variables are not capable of solution to any great extent. An example or two will suffice to illustrate our process.

Ex. 11. $$\Delta_x u_{x,y} - \Delta_y u_{x,y} = b.$$

The solution is $$u_{x,y} = \frac{1}{\Delta_x - \Delta_y} b$$

Treat Δ_y as a constant c, then the solution of $(\Delta_x - c) u_{x,y} = b$ is

$$u_{x,y} = (c+1)^x A - \frac{b}{c}$$

$$u_{x,y} = (\Delta_y + 1)^x v_y - \Delta_{y}^{-1} b$$

$$= \left(e^{\frac{d}{dy}}\right)^x v_y - by$$

$$= v_{y+x} - by$$

v_{y+x} being an arbitrary function of $y+x$.

Ex. 12 $\Delta_x^{\frac{1}{2}} u_{x,y} - \Delta_y^{\frac{1}{2}} u_{x,y} = b$

This equation gives $u_{x,y} = \dfrac{1}{\Delta_x^{\frac{1}{2}} - \Delta_y^{\frac{1}{2}}} b$

$$= \frac{\Delta^{\frac{1}{2}}{}_x + \Delta_y^{\frac{1}{2}}}{\Delta_x - \Delta_y} b$$

$$= (\Delta^{\frac{1}{2}}{}_x + \Delta^{\frac{1}{2}}{}_y)(v_{y+x} - by) \qquad \text{(Ex. 11.)}$$

We have thus succeeded in solving equations with fractional indices of all forms corresponding with the ordinary forms of Linear Differential Equations, whether total or partial,—whether solitary or simultaneous. We have also placed the Calculus of Fractional Differences on the same footing with respect to the ordinary Calculus of Fractional Differences as that which the Calculus of General Differentiation occupies relatively to the ordinary Differential Calculus.

P. KELLAND.

EDINBURGH, *October* 10, 1846.

XXI.—*Observations on the Principle of Vital Affinity, as illustrated by recent discoveries in Organic Chemistry.* By William Pulteney Alison, M.D., F.R.S.E., *Professor of the Practice of Medicine in the University of Edinburgh.*

(Read 1st and 15th February 1847.)

Part II.

It may be remembered that, in the paper formerly laid before this Society on this subject, I endeavoured to establish the principle still disputed by some physiologists, that the laws which regulate the chemical relations, as well as those which regulate the visible movements of the particles of matter, undergo a certain determinate modification or change in living bodies, which is essential to the commencement and to the maintenance of the organization of those bodies; and farther, that I undertook the task of attempting to deduce, from the numerous but somewhat discordant experiments and observations lately made on the subject, certain inferences which appear to be well ascertained, although not generally admitted, as to the essential nature of this change, *i. e.*, as to laws which regulate those chemical actions which are peculiar to the state of life, and essential to the maintenance of organization, both in vegetables and animals.

In confirmation of my statement of the general principle of Vital Affinity, as distinguished from simply chemical affinities, I have much satisfaction in quoting two sentences from the last edition of Liebig's "Animal Chemistry." Some of the statements of general principles made by this author, seem to me open to objection, and some I do not profess to understand; but the following is simple and precise; and, considering the authority of Liebig as a chemist, may, I think, be held nearly decisive as to the soundness of the principle. "A *fundamental error*, committed by some physiologists is, that they suppose the chemical and physical forces alone, or in combination with anatomy, sufficient to explain the phenomena of vitality. It is, indeed, difficult to understand how the chemist, who is intimately acquainted with chemical forces, should recognise in the living body the existence of *new laws*, of new causes, while the physiologist, who is little or not at all familiar with the action and nature of chemical and physical forces, should think himself ready to explain the same processes with the aid of the laws of inorganic nature alone."—*Animal Chemistry* (*third edition*, p. 252.)

The first and most fundamental of these general principles (likewise considered in my former paper) is the power of vegetable life, under the influence of light, to decompose the carbonic acid existing in the atmosphere,—set the oxygen free, fix the carbon, and form with it and the elements of water, starch, sugar, gum, and the analogous compounds. Our knowledge of this power, of the effects

which result from it, and of the period when it must have been first exerted on the earth's surface, enables us to assert with confidence, that by means of it, the whole organised creation has been, as DUMAS expresses it, the offspring of the air; and that it was by enabling the rays of the sun to excite this action in certain particles of matter, existing in the atmosphere, but destined to be either the first specimens, or the first germs of vegetable life, that "a beneficent God," to use the striking expression of LAVOISIER, "has strewed the surface of the earth, first with organized structures, and then with sensation and thought."

In proceeding farther to inquire into the laws of Vital Affinity, we must always keep in mind the general arrangement or classification, long ago made by Dr PROUT, of all the organic compounds, of which any organized structures, vegetable or animal, are composed, into three groups or classes, the Saccharine or amylaceous, the Oily, and the Albuminous; and the important observation, I believe first made by him, that the food of most animals contains all these compounds, and that no complex animal structure can be maintained without the concurrence of at least two of these kinds of compounds in its food.

I do not think it is going too far to say that we have now a general knowledge of the laws or conditions under which all these compounds are formed in living bodies, taking the starch formed from carbonic acid and water as the foundation of all. But we perceive farther, that that these laws, *varying* in *different parts of the same structure*, and at *different times in the same parts*, and being of *transient duration* in all, are liable to an *influence of time and of place*, and in animals to a farther influence of mental changes, which is quite analogous to the vital actions, both of muscular and nervous organs, but is strongly contrasted with the uniformity of the laws that determine the changes of inorganic matter. And if this be so, we may assert that considerable progress has been made, both in establishing and in illustrating the doctrine of vital affinity, as a first principle in physiology.

I. The formation of Oil or Fat in living bodies is, perhaps, that part of the chemical processes there carried on, which is now the best understood, and the study of which gives us the clearest insight into the nature of vital affinities. We need not enter into any of the simply *chemical* questions as to the mode of combination of the fatty acids and bases in the different kinds of fat; it is sufficient for our purpose to observe that they are found very generally, though very variously disposed, in almost all vegetables and animals, and even in the earliest stages of their existence; the store of nourishment contained in the seed and in the egg, containing a proportion of fatty matter. And though there is considerable variety in the different kinds of fat or oil, they all differ from the varieties of starch, by having a much smaller proportion of oxygen, and, of course, a larger proportion

of carbon and hydrogen. The composition of most fats is stated by Liebig to be $C_{12} H_{10} O_1$; and we have thus, therefore, another compound formed apparently by vital affinity, indicating a peculiar attraction of the two first elements for one another, and a feeble attraction for oxygen. Indeed, in the composition of wax (one of this family of compounds), as stated by MULDER, the proportion of oxygen is only one equivalent to 24 of carbon; in cholesterine, the proportion of carbon to oxygen is stated as high as 36 to 1; and in many volatile oils, no oxygen has been detected.

Supposing such a peculiar affinity to act, there is obviously no difficulty (on looking at the numbers indicating the proportions of the elements) in understanding the formation of these compounds out of starch ($C_{12} H_{10} O_{10}$), just as there is none in understanding the formation of starch or sugar (although by an affinity occurring only in living bodies, and which we regard as vital) from carbonic acid and water ($CO_2 + HO$), in living vegetables, where a continual evolution of oxygen attends the growth; particularly if we suppose that the carbonic acid taken in by the leaves and roots, is carried to, and decomposed in, all parts of the plant: the formation of the fatty compounds, is, no doubt, one of the processes by which the oxygen is set free. But in the case of animals, where (with the exception of some of the infusory tribes) there is no evolution of oxygen, the formation of fat from starch presents a difficulty. Yet the numerous observations and experiments of LIEBIG and of CHEVREUL and MILNE-EDWARDS, leave no room for doubt that various animals, fed chiefly on varieties of starch, or bees fed on sugar, form a much larger quantity of fat, oil, or wax, than they have received mixed with their food, and this when they are exhaling no pure oxygen, but, on the contrary, compounds of hydrogen and carbon with oxygen, viz., water and carbonic acid. Indeed, Dr ROBERT THOMSON having ascertained by repeated experiments, that the quantity of butter yielded by cows bears no fixed proportion to the quantity of oleaginous matter contained in their food, varying indeed from one quarter to nearly the whole of the oleaginous ingesta, thinks himself justified in inferring that "the butter cannot be supplied from the oil of the food." (*On the Food of Animals*, p. 156.)

It is quite certain that in this action, in all animal bodies, the greater part of the oxygen of the starch employed must unite with a portion of its carbon and hydrogen, and pass off in the excretions just noticed, leaving the small remainder of the oxygen in combination with the predominant quantities of carbon and hydrogen.

It appears possible, indeed, that *all* the oxygen which must be separated from starch before it can be converted into fat, may be evolved in combination with part of the carbon and hydrogen of the starch, without any constituent of the air taking any part in the process; but the quantity of fat formed would then be small, and it is also possible that the oxygen of the air may be concerned

in the metamorphoses to which starch is liable in a living body; and as we know the importance of oxygen in maintaining (in one way or other) all vital action, the latter supposition is more probable.

If, *e. g.*, we suppose 4 atoms of starch to yield 2 of fat, we must subtract from

	48 C	40 H	40 O
	24 C	20 H	2 O
leaving	24 C	20 H	38 O = 20 HO 9 CO^2 + 15 C;*

so that on this supposition 15 atoms of carbon are set free, and as these do not appear, they must unite with the oxygen of the air, and take the form of carbonic acid; and then the fat which appears, together with the water and carbonic acid thrown off, will account for all the elements concerned in the action. In this process, therefore, supposing the quantities of starch taken in, and of fat formed to be as above, 30 equivalents of oxygen must be absorbed; so that we perceive the use of oxygen in the change, and the necessity of its presence, although the fat formed contains so much less oxygen than the starch.

That this should be the real nature of the change is just what we ought to expect, if, agreeably to the supposition formerly made, the starch taken into the blood of a living animal, is acted on at certain parts of the body by two powers, and divides itself between them, viz., a vital affinity, in which carbon is the chief agent, which leads to the formation of fat, and the simply chemical affinities, exerted chiefly by oxygen (continually taken into the blood), by which, if removed from the living body, we know that it would gradually be resolved into carbonic acid and water. And that this is the real state of the case we are fully assured by a simple but very important observation, viz., attending to the effect of *exercise* on the formation or deposition of fat in the living animal body. As we see by the numbers given above, that a certain amount of oxygen must be absorbed, and a certain quantity of carbonic acid and water, formed by its help, must be excreted, to enable starch to yield oil or fat by the process there represented, we can understand that moderate exercise should favour the change; but when exercise is carried beyond a very moderate extent, we know that the circulation and respiration being much accelerated, and the quantity of oxygen taken into the living blood being much increased, the effect is, to increase the exhalation of carbonic acid and water, and proportionally to diminish the deposition of fat; *i. e.*, to give a preponderance to the simply chemical affinities exerted by the oxygen, over the vital affinity, which would tend to the formation of fat.

From this simple fact we may infer, 1. That the vital affinity by which oil is

* It need hardly be said, that all these numbers are given, not as indicating the exact changes which take place when the organic compounds are formed, but only as illustrating their general nature.

formed from starch, or by which its elements are held together, does not supersede its natural chemical relations, but only adds a new chemical power to those which can operate on it, and allows of a division of the starch between the result of a vital and a simply chemical affinity; and, 2. That the vital action by which fat is formed or maintained, is of no great strength, as compared with the simply chemical affinities to which the same matter is liable; being superseded simply by an increased supply of oxygen. And we cannot doubt that, in this as in other vital chemical processes, the oxygen, although not taken into the organic compound formed, aids its formation materially, by promoting, on the principle of divellent affinity, the other parts of the metamorphoses whereby it is produced. We shall see afterwards the importance of having it established by this simple example, that the oxygen of the air, when taken in full quantity into the blood, is capable of combining, somewhere in the course of the circulation with a part of that carbon and hydrogen, recently absorbed into the blood, which, under a smaller supply of oxygen, would form a living texture; and that the combination of these portions of the ingesta with oxygen, are one source of the excretions.

There are other facts which lead to the same conclusion, as to the affinity by which fat is formed, being more nearly akin than most vital actions to simply chemical affinities; particularly,—

1. The formation of Adipocere, not from starch, but from albumen, after vitality is over, when undergoing decomposition under ground, where there is a full supply of water and but little air, so that the supply of oxygen is less than in ordinary putrefaction, which may be understood thus:—

	C	N	H	O	
	48	6	36	14	= Albumen
Add			12	12	Water
				1	Oxygen
	48	6	48	27	
Subtract	36		30	3	Fat
	12	6	18	24	= Carbonic Acid and Ammonia

which escape, and the attraction of which for each other, no doubt in part determines the result.

2. Again, in the living body, but in a feeble constitution, along with great emaciation, and a deficient supply of oxygen, a morbid deposition of fat sometimes takes place, in circumstances where it could not have been anticipated, but only in particular parts. Some distinct cases of this kind have lately attracted attention, one in the kidneys, in one form of BRIGHT's disease, another in the liver, as in many phthisical cases, and a third in the atheromatous exudations

so common on the arteries. It may be suspected that in these cases the formation of fat is by an affinity hardly more vital than the formation of adipocere,—in both cases the decomposition of albumen to form the fat, being aided by the simply chemical affinities, of carbon for oxygen, and of hydrogen for azote.

3. The same peculiarity of the attractions by which fat is formed in the animal economy may be admitted in explanation of the more general fact, that in a healthy constitution, when more, particularly of amylaceous, food is taken than is required for the nutrition of the more important textures, and when little oxygen is taken in, the excess always tends to the deposition of fat, which implies that a large portion of the oxygen of that food has gone off as carbonic acid and water.

The process of the formation of oil from starch in the animal body, admits of an instructive comparison with the simply chemical one of the formation of alcohol from the same matter,—at least, from a compound fluid of which starch (first converted into sugar by the kind of fermentation formerly mentioned) is the chief constituent, in fermentation; *e. g.*, the changes in the vinous fermentation of grape-sugar, are represented thus,—

$$C_{12}\,H_{14}\,O_{14} = 2\,(C_4\,H_6\,O_2) + 4\,CO_2 + 2\,HO,$$

that is, the elements of grape-sugar resolve themselves into two equivalents of alcohol, four of carbonic acid, and two of water. In this case, as in the formation of fat, the starch or sugar is divided into three parts, water, carbonic acid, and a peculiar compound fluid. In both cases, the oxygen of the air is necessary to the commencement, and probably to the continuance, of the process, although in both, the new compound formed contains less oxygen than the starch or sugar from which it is produced. In both cases, a third body is present, and its influence somehow promotes the process, besides the oxygen and the starch, viz., in the one case, yeast, or some kind of ferment, itself in a state of decomposition, which it imparts, without giving up any part of its substance, to the starch or sugar; in the other case, a living cell, composed of gelatin, which is itself undergoing a simultaneous change, by a living process. In both cases, extension of the change takes place, as from a centre, from this third body, through the fluid in which the change commences. In both cases, the compound formed is not stable; and the portions of the starch which go to form it are destined ultimately to follow the same course as those portions which are resolved into carbonic acid and water. In the one case, the compound formed, $C_4\,H_6\,O_2$, contains a less proportion of carbon than any of those which we regard as endowed with strictly vital properties; while, in the other, the compound formed, $C_{12}\,H_{10}\,O_1$, has the characteristic predominance of carbon. But if we are asked, Why we regard the one as the result of a simply chemical process, and the other of a vital affinity? I apprehend the sufficient answer to be,

that the one is a change which uniformly results when the sugar is exposed to the influence of air, water, and a certain temperature; and is in contact with a substance undergoing some part of that decomposition and chemical change to which living bodies are liable after the phenomena of life are over; whereas the other is not seen, in the presence of those substances, and under those conditions as to air, water, and temperature, in which it here takes place, unless the starch is at the same time in contact with living cells,—*i. e.*, cells forming a part of a body in which the peculiar phenomena of life are then exhibited.*

II. The next question is as to the formation of the Albuminous, or what have lately been called the Protein, compounds in animal bodies. The late acrimonious dispute as to the existence of Protein, should rather be termed a dispute as to the exact composition of the compound to which MULDER gave that name, and which is thrown down from the solution of either albumen or fibrin, in potash, by acetic acid. Of the precipitate being the same in both cases there is no doubt; and we shall avoid the controversy entirely, by using the term Albuminous Compounds, as Dr PROUT did, instead of the term Protein.

Since it has been clearly ascertained, that the vegetable gluten is identical in composition with the albuminous compounds,—*i. e.*, fibrin, albumen, and casein of animals,—no doubt can exist that the formation of a great part of the albumen found in animal bodies must take place in vegetables; and, I presume, it is also generally agreed that the chief agents in this farther change, beyond the formation of starch and of fat, are sulphur, and ammonia or its elements, taken into the fluids of the vegetable, although it is still doubtful from what sources this ammonia or its elements may be originally derived, and particularly whether, in any circumstances, the azote of the atmosphere is concerned in producing it.

Some experiments recorded by DUMAS,† however, seem to leave no room for doubt, that certain families of plants, in one way or another, fix azote from the air, being found to add largely to that contained in their seeds, when germinating and growing merely in silica and water; and it is by no means ascertained, that this azote passes necessarily into the state of ammonia before it is applied to the nourishment of those vegetables. And the statements of MULDER seem equally conclusive as to the fact, that ammonia may be

* It is no objection to this statement, that oily matters may, in different cases besides that of adipocere already noticed, be formed from organic compounds in the dead state, *i. e.*, by simply chemical affinities. To establish that the affinity by which it is formed in a living structure is vital, it is not necessary to shew that oil cannot be formed, under any circumstances, by simply chemical laws, but only to assure ourselves, that it cannot be formed by those laws from the substances, and in the circumstances, in which it is continually formed in certain living cells.

† Balance of Organic Nature, p. 77.

formed by the union of azote from the atmosphere with hydrogen from water, whenever another substance, exerting an attraction for the oxygen of the water, is present.—(*Chemistry of Vegetable and Animal Physiology*, p. 149, *et seq.*) Now, as carbonic acid and water form starch, or its allied compounds, in the living vegetable, by the attraction of carbon for the elements of water, to the exclusion of oxygen; and as the starch then forms oil, by the attraction of the carbon to hydrogen, to the exclusion of great part of the remaining oxygen; so, on the introduction of ammonia, or its elements in a state fit for entering into new combinations, into the scene of those metamorphoses, it is only in accordance with what we know of the nature of these vital affinities, to suppose that the carbon may attach to itself the elements of this ammonia, to the exclusion of the elements of water and of oxygen, matters which are known to be continually thrown off by vegetables, during the continuance of these vital processes. Thus we have the elements of starch, 48 C, 40 H, 40 O *plus* the elements of ammonia, 6 N, 18 H, = 48 C, 6 N, 58 H, 40 O = 48 C, 6 N, 36 H, 14 O (the elements of albumen) *plus* 22 H O + 4 O, a considerable quantity of the water, and a small quantity of the oxygen, which are continually exhaled by the plant.

Thus, during the whole process of the formation of organic compounds in the vegetable, we see that the vital affinities shew themselves by the attractions of Carbon, first for the elements of water in preference to oxygen, then, either for the hydrogen of those elements, in preference to the oxygen, or for the elements of water, with an excess of hydrogen, along with those of ammonia; and thus, by these peculiarities of attraction of Carbon, for the elements of water, for hydrogen, and for azote,—to the more or less complete exclusion of oxygen,—we see that the essential materials of all organized matters may be easily formed,* while water and oxygen, the known excretions of vegetables, only escape.

The point at this moment most disputed, and the settlement of which is most essential to the precise comprehension of the nature of vital affinities, is, Whether there is any formation of albuminous matter in *animal* bodies? and it is obvious, that there is a difficulty in regard to its formation from starch, just similar to that which was stated as to the formation of oil in the animal body, because we see no evolution of oxygen; but it is also certain that this may be got over, precisely in like manner as in the former case, by supposing—what is quite in accordance with known facts—that a considerable absorption of the oxygen of the air attends the process, and that, with its help, a large portion of the carbon

* This may be shortly stated thus $CO_2 + HO$ = Carbonic acid and water. From this is formed,
$C + H + O$ = Sugar, oxygen going off. From this,
$C_{48}\,H_{40}\,O_{40}$ = Starch, water going off. From this, either
$C_{48}\,H_{40}\,O_4$ = Fat, oxygen going off. Or,
$C_{48}\,N_6\,H_3{}^6\,O_{14}$ = Albumen, ammonia being added, and water and a little oxygen going off.

and hydrogen are thrown off in carbonic acid and water. Thus, supposing a large quantity of starch, 60 C, 50 H, 50 O, to unite with a small quantity of ammonia, we have

	C	N	H	O
	60	6	68	50 and adding 20 of oxygen,
we have	60	6	68	70=48 C, 6 N, 36 H, 14 O,

(the elements of albumen) + 32 HO + 12 CO_2, the water and carbonic acid which escape. Or, adding an equivalent of oil, we may have

	C	N	H	O	
	48	...	40	40	Starch.
Add	12	...	10	1	Oil.
	...	6	18	...	Ammonia.
	60	6	68	41	
Subtract	48	6	36	14	Albumen.
	12	...	32	27	Adding 29 oxygen,

we get 12 CO_2 + 32 HO carbonic acid and water.

It is certain, therefore, that if the elements of ammonia can be set free in the primæ viæ of an animal, starch absorbed from thence, with or without the addition of oil, may be converted into albumen in its blood, without any other matter being thrown off than the water and carbonic acid, which undoubtedly escape from every animal. If this be so, we have here another division of the elements of the ingesta, between substances exerting a vital and a simply chemical affinity for them, and another formation of part of the excretions, by the help of the oxygen of the air, from matters recently absorbed, and which aid in the nourishment of the animal. But whether this is a process that actually goes on in the animal economy, or whether all the albuminous compounds of animal bodies have passed into them, directly or indirectly (but ready formed), from vegetables, is the point at this moment the most important to be ascertained.

As it is obvious that the albuminous compounds, and the gelatinous compounds (which are closely related to them, and are generally thought to be formed from them), compose the greater part of the animal textures, and are equally the groundwork of all animal structure, as starch is of vegetables, this inquiry involves the essential point of distinction, so far as chemistry goes, between vegetables and animals. It is well known that both LIEBIG and DUMAS have expressed a decided opinion that no albumen is formed in animals; and the latter author has contrasted, in a lively manner, vegetable and animal life in this respect, representing the former as always a reducing or deoxidating apparatus, and the latter as an apparatus of oxidation or combustion, *i. e.*, of the destruction, never of the formation, of any organic compound. But he does not appear to have adverted particularly to the question which seems to me the most essential in a physiological view, viz., what are the chemical changes during the state of life, whether

in vegetables or animals, which are distinctly at variance with the ordinary laws of chemistry, and which we must therefore ascribe to vital affinities?

It is evident that what, in physiological language, is commonly called Assimilation, includes two distinct actions, both, in many cases, as I believe, strictly vital; *first*, the mere selection and attraction of a part of a compound fluid, to be added to a living body; and, *secondly*, the *transformation* of the elements of two or more compounds, so as to form a new compound, similar to one already existing in the living body wherein this change occurs. If DUMAS's view of the subject were to be adopted, we should say that animals can exert only the first of these powers, the simple selection and attraction of one of the ingredients of a compound fluid by each organ or texture, without any power of *transformation*, or formation of new compounds; and accordingly, he says that "it is in plants that the true laboratory of organic chemistry resides."

But if we state the proposition thus generally, we may state various facts to shew, that it is incorrect. It is quite certain, as already stated, that oil or fat may be *formed* in animal bodies, by a new arrangement of the elements of starch, attended by an evolution of much of its oxygen, and of part of its carbon and hydrogen, effected by the aid of the oxygen of the air; and the influence (already noticed) of exercise, *i. e.*, of an increased application of oxygen, on this change, shews distinctly that the recent ingesta are liable to two influences in a living animal, one of which is an action of oxidation or combustion, throwing off water and carbonic acid, but the other is strictly an action of reduction, by which a quantity of oxygen is separated from its combinations in an organic compound, while a fresh compound, constituting part of the animal frame, is formed. And the fat of the animal body, which may be thus formed, is not to be considered as a merely unorganized appendage to the textures. It appears from some of LIEBIG's observations, that the muscular flesh of all animals, after being cleared of all visible fat, still retains a considerable and variable quantity in its substance; and we know that in two of the most important textures of the body, nervous matter and bone, fat is an essential ingredient.

In like manner, the formation of the essential ingredient Gelatin in the animal body is the result of a new arrangement of elements, attended with evolution of carbon and hydrogen, by the aid of the oxygen of the air, but probably not with absorption of oxygen.

In the case of Inflammation, we see distinctly that, in connection with an increased action of nutrition or deposition of plastic lymph, there is a transformation of portions of the blood to form the compound, very similar to gelatin, termed, by MULDER, the Tritoxide of Protein, which is found there in very unusual quantity; and in other morbid actions, in certain chronic malignant diseases, we see compounds, altogether foreign to the natural organization, formed and even rapidly extended; the formation of which is certainly neither a simply chemical act

of oxidation, nor a mere selection and appropriation of compounds previously formed in vegetables.

On the other hand, it is known that there is an evolution of carbonic acid as well as water from vegetables,—from the parts of fructification during their development even in the day time, and from all parts during the night; and it appears quite possible that, in both cases, this may be by a process of slow combustion, similar to the process of oxidation which DUMAS considers as characteristic of animal life only. For, although it has been stated by DUMAS that the carbonic acid given up by vegetables during the night is only what has been absorbed by their roots, and passed unchanged through their substance, yet I do not find any distinct proof of this in his writings. It is certainly true, that the organic compounds formed by vegetables, and taken into animal bodies, ultimately undergo in them a chemical change nearly equivalent to slow combustion, and are thus returned to the inorganic world; but this is in the processes of absorption, decomposition, and excretion, of the animal textures, to be considered presently; and this fact affords of itself no proof, that in the previous growth and development of animal textures, there may not be an actual formation of albuminous compounds, as well as of gelatin and fat.

These facts appear sufficient to shew, that there is no such direct opposition between vegetables and animals, as to the chemical results of their vital action, as DUMAS has represented; and even to make it probable, that, during the organic or vegetative life of animals, there will be a formation of albuminous matter, equally as of gelatin and fat.

In fact, this question can be only finally decided by experiments to shew whether or not the whole quantity of albumen deposited in the textures of a growing animal may be greater than that contained in its food; or whether the azote excreted, during a pretty long period, from an animal, by the bowels, kidneys, skin,* and lungs† (for it appears to be well ascertained, that, from all these parts, there is a frequent, if not an habitual, excretion of azote), is greater, under any circumstances, than the quantity of that element contained in the albuminous portion of its food, which is the only ascertained channel of the introduction of azote into the animal system; and, although this is a difficult inquiry, we cannot suppose that the difficulties are insurmountable. If such an excess of excretion of azote shall be ascertained, it will be nearly enough to entitle us to conclude that albuminous matters can be formed in the animal body, and yield it during their decomposition there. It is not enough to say, that there is no occasion for more azote in the animal economy than is contained in the albuminous ingesta, because what is there contained is already in just the same proportion to

* See GOLDING BIRD on Urinary Deposits, p. 104.

† See DU LONG, quoted by DUMAS (Organic Nature, p. 106).

carbon and hydrogen, as that which exists in the blood, or in the textures of animals. As there is, in the whole of the ingesta of animals, a great excess of carbon and hydrogen over their proportion to azote in albumen, and as oxygen is always present in the blood, it is quite possible that a part of the azote of the albumen taken in, may be thrown off in combination with portions of those other elements, by the bowels and kidneys, without entering into the textures; and that the nourishment of the textures may be in part due to fresh albumen, formed in the animal body by help of oxygen from the lungs, and of azote taken in by another channel; just as we are nearly sure that part of the oil taken into an animal is often decomposed and thrown off, and that fresh fat is often formed from the starch or sugar of the ingesta.

There is one mode, pointed out by LIEBIG, in which we can have no doubt that azote must be introduced into the blood of animals, independently of the albuminous ingesta, viz., by the air which is contained in the water, and still more in the saliva, continually taken into the stomach. "During the mastication of the food, there is secreted into the mouth, from organs specially destined to this function, a fluid, the saliva, which possesses the remarkable property of inclosing air in the shape of froth, in a far higher degree than even soap-suds. This air, by means of the saliva, reaches the stomach with the food, and there its oxygen enters into combination, while its nitrogen is given out through the skin and lungs."*

Now, what proof is there that the azote, thus believed to be set free in the stomach, is excreted, unchanged, by the skin and lungs? Is it not much more probable that it enters into fresh combinations in the primæ viæ and in the blood, and is only separated from the blood, when, by the agency of the oxygen of the air, acting, under the circumstances to be afterwards stated, with peculiar energy on some of the constituents of the blood, it is disjoined from its union with carbon and hydrogen.

In fact, the azote thus set at liberty in the stomach, must be in circumstances almost exactly similar to those in which, according to the statements of MULDER and others, ammonia is formed from air, even by the help of inorganic matter; still more when organic matter, although non-azotised, is present in a state of decomposition, or an analogous condition.† "By all porous substances ammonia is produced,—provided they are moist, are filled with atmospheric air, and are exposed to a certain temperature."

"When reddened litmus paper is hung up in a bottle, filled with pure atmospheric air, and when pure iron-filings, moistened with pure water, are laid at the bottom, then the red litmus is quickly turned blue by the action of ammonia, formed from the nitrogen on the air, and the hydrogen of the decomposed water, the oxygen of which had combined with the iron.

* LIEBIG'S Animal Chemistry, pp. 113-4.

† MULDER, p. 149, *et seq.*

"Such a formation of ammonia continually takes place in the soil. There, atmospheric air is present, and consequently nitrogen; hydrogen is continually liberated, and thus the conditions necessary to the formation of ammonia are fulfilled as often as cellulose, ligneous matter, starch, &c., are changed either into humic acid, or into other constituents of the soil."

That a partial decomposition of organized matter takes place in the stomach, and is, indeed, the first part of the changes occurring during digestion, seems to be sufficiently proved by some curious and important observations of LIEBIG himself.* "The fresh lining membrane of the stomach of a calf, digested with weak muriatic acid, gives to this fluid no power of dissolving boiled flesh or coagulated white of egg" (the supposed property of the Pepsin, or extract of the mucous membrane there.) "But if previously allowed to dry, or if left for a time in water, it then yields, to water acidulated with muriatic acid, a substance in minute quantity, the decomposition of which is already commenced, and is completed in the solution. If coagulated albumen be placed in this solution, the state of decomposition is communicated to it, first at the edges, which become translucent, pass into a mucilage, and finally dissolve. The same change gradually affects the whole mass, and, at last, it is entirely dissolved."

I think we cannot doubt, therefore, that the air introduced into the stomach of animals, and decomposed there, as LIEBIG supposes, must be in circumstances peculiarly well adapted for the generation of ammonia, or the setting free of its elements; which, as we have seen, is all that appears necessary to explain the gradual formation in the matters absorbed from the stomach, of albumen out of non-azotised ingesta; under the influence of vital affinities, similar to those by which albumen is formed in vegetables.

I am aware that LIEBIG states with confidence that experiments prove that the whole of the azote excreted in a given time by an animal, is not more than that which is taken in by its albuminous ingesta; but in this he relies chiefly on the experiments of BOUSSINGAULT, and these experiments are not considered by the author himself as altogether satisfactory; nor can they be satisfactory without farther investigation of the quantity excreted by the skin and lungs, into which he did not inquire. (See *Dumas*, p. 106.)

I admit it to be certain, however, from a simple comparison of the quantities of albuminous ingesta and the azotised excretions, that the formation of albumen in animals can be to no great extent; and I am clearly of opinion that the distinction drawn by LIEBIG, of the azotised and non-azotised ingesta of animals, and the evidence he has given of the chief destination and use of each, constitute the most important improvement lately made, in this department of physiology. It appears now ascertained; 1*st*, That the latter class of aliments are incapable, *in*

* Animal Chemistry, pp. 110-1.

themselves, of adding to any of the animal textures except the fat; but that they are the chief material on which the oxygen of the air acts to keep up the animal heat. 2*d*, That the main reliance of the animal body for the nourishment of all its parts must be on the former class of aliments; their adequacy for that purpose being beautifully exemplified in the life of the chick *in ovo*, where all the textures are formed out of the albumen, partially converted into gelatin in the process, and with the addition of a small quantity of oil from the yolk; the oxygen of the air being essential to the vital movement, but no farther concerned in the results, than as it carries off a certain portion of the carbon and hydrogen from the moving matter, and so occasions a loss of substance during the process of incubation. 3*d*, That the azotised ingesta, or the textures formed from them, are themselves liable to this action of the oxygen when the non-azotised ingesta are deficient; and, therefore, that an important use of the non-azotised food is, to protect the albuminous constituents of the blood and the animal textures, from an influence of the oxygen of the air, which, but for that protection, would be injurious, and ultimately destructive. And I may perhaps be allowed to state what seem to me the most important results, both as to Physiology and Pathology, which are involved in these principles.

1. Our ideas of the use of the digestive apparatus of animals are rendered much more simple and precise. I have stated, indeed, that DUMAS appears to have erred in the way of extreme simplification, when he says that "an animal only assimilates" (*i. e.* selects and attracts) "organic structures already formed; that he forms none;" that "digestion is therefore a simple process of absorption, soluble substances passing directly into the blood (*i. e.* by the veins), for the most part without alteration, and insoluble substances making their way into the chyle after having been sufficiently comminuted, to be imbibed by the lacteals." But although we suppose that certain transformations, as well as simple absorption, must be commenced, at least, in the digestive organs, we are sure that no complication of apparatus is necessary for accomplishing them; the most important of all transformations necessary to life taking place in vegetables, and in organs of extreme simplicity.

The following may be stated as the purposes which are served by the digestive apparatus of every kind of animal, whether carnivorous or herbivorous, and the greater complexity of the arrangements in the latter tribes must be considered as intended merely to present a larger surface, and afford a longer time, for the accomplishment of changes which are, in fact, identical in kind, and all of which may be effected in the simplest form of apparatus.

(1.) This apparatus is obviously necessary, as stated by CUVIER, for the support of textures whose vital action is dependent on a continuous supply of nourishment, to afford that continuous supply from aliments, the reception of which, in the case of animals, is only occasional, and sometimes long delayed.

(2.) It is useful, as providing for the separation and immediate expulsion

from the body of those parts of the ingesta which are wholly inapplicable to nutrition, and for which no part of the living structure has any vital attraction.

(3.) It is especially useful, as giving the necessary fluidity to aliments which must be moved to all parts of the animal frame, and applied to the nourishment of the organs in a state of minute subdivision, but which are often introduced into the system in a solid form, having been formed in one living structure, vegetable or animal, and applied to the purpose of nutrition in another, and often after a long interval of time. For this purpose, it appears certain, that various contrivances are employed: in many cases, the mechanical process of attrition is an essential preliminary; in all cases, water is employed; in most cases, it would appear, especially from the observations of LIEBIG, that a certain degree of incipient decomposition— speedily arrested by the action of vital affinities, but beginning on the mucous membrane, and extending to the mass of aliments—precedes and aids the action of the solvent; just in like manner as an incipient decomposition of starch, and formation of soluble sugar, precedes the development of vegetable shoots and flowers; but especially the requisite fluidity is given by solvents, applied at different spots, and which are prepared from the blood, under the influence of appropriate stimuli, by a vital attraction, or selecting power, existing at those parts. Thus, an acid liquor is prepared at the stomach and at the cæcum, and, with a similar intention, according to recent observations, it would appear, that an alkaline liquor is prepared in the salivary glands, liver, and pancreas.

(4.) The most soluble part of the ingesta, and especially the amylaceous portion, must necessarily be taken up by the veins, and carried directly to the liver to form bile; and as this portion, unless combined with azotised matter, is inapplicable to the nutrition of any texture except the fat, we see here one ground for the opinion to be afterwards stated, that the animal matter of the bile is chiefly useful as a part of the provision for the agency of oxygen, and the maintenance of animal heat.

(5.) Although we are uncertain how far transformations of the organic compounds are effected in the animal economy, as preliminary to nutrition, yet we have seen that some such transformations must be admitted as a part of the living power of animals, for the formation of fat, of gelatin, perhaps also of albumen; and this process is pretty certainly commenced in the chyme, in the primæ viæ, and particularly in the organized globules there formed, to be afterwards carried on in the course of the circulation.

2. In the next place, the principles laid down by LIEBIG as to the distinction between the azotised and non-azotised classes of aliments, enable us distinctly to understand the law of PROUT, as to the necessity of a mixture of at least two of the three kinds of aliment which he distinguished, the albuminous, oily, and saccharine, in order to maintain life. In fact, I have no doubt we may go farther (in

consequence of the discoveries made as to the existence of albuminous matter in vegetables since Dr PROUT wrote), and assert that more or less of albuminous matter is always necessary, because it alone, of all the solid or fluid ingesta, contains the azote which is a necessary constituent of animal textures; and that it must be combined either with starch or with oil, or both; partly because oil is an essential constituent of parts of the body, and must either be furnished ready made, or formed in the body from starch; and partly because the animal heat, the first requisite of vitality, can only be maintained by the oxygen of the air combining with carbon and hydrogen in the blood; and if it does not find these elements in sufficient quantity, and in a fit state for such union, in the other constituents of the blood or of the textures, it will attack the albuminous portions of the blood and textures, and so cause decomposition and wasting of the body.

We see likewise the importance of oily food, which, containing the largest proportion of carbon and hydrogen, will yield to the oxygen the largest quantity of carbonic acid and water, and therefore evolve the greatest quantity of caloric,—in cold climates; and of saccharine and amylaceous food which, containing more oxygen in itself, will furnish a smaller quantity of calorific compound with the oxygen of the air,—in warm climates; particularly as the supply of heat from this kind of ingesta is farther regulated and moderated by the action of the liver, in a way to be afterwards considered.

3. We understand the principle, on which the wasting of the body is effected, either in cases of denial of aliments, or of disease preventing their reception or digestion; *i. e.*, we understand that the oxygen of the air, introduced regularly and uniformly in the blood by respiration, but meeting there with very different compounds as the privation of ingesta continues, is the main agent in the process; acting first, as it must do in the healthy state, on the non-azotised compounds existing in the blood, oil, cholesterine, or other constituents of the bile, and starch, or matters recently formed from starch, and nearly destitute of azote, and which readily give up their carbon and hydrogen; next acting on the non-azotised portion of the solid textures, *i. e.*, the fat, and causing emaciation; afterwards acting on the albuminous portions of the blood itself, rendering it more serous; and then acting directly or indirectly on the solid textures, determining ultimately such absorption of the substance of the brain and nerves as causes delirium and insensibility, and such absorption of the muscular textures, as causes death by asthenia. It can only be by successively acting on these different matters, that the oxygen can find the quantity of carbon and hydrogen with which it must unite in the course of the circulation, to account for its own disappearance and for the quantity of carbonic acid which is known to be still thrown off, for days and weeks, while no carbonaceous matter is added to the blood; and the order in which the successive changes on the sensible qualities and functions of the body occur, corresponds perfectly with the belief that the oxygen, acting on the dif-

ferent parts more or less rapidly, as they give up their carbon more or less easily, is the immediate agent by which the extenuation of all is effected.

4. We understand, certainly not completely, but better now than formerly, the nature of the changes which take place in animals long fed on one kind, even of albuminous food, equally as when albumen is withheld; and which appear in both cases to indicate a deficiency of the albuminous constituents of the blood; and likewise, certain phenomena in disease, connected with deficiency of those albuminous constituents.

There are several facts connected with such diseases which we cannot understand, until we have some farther information as to the relation to each other in the living body, of the different constituents of the blood which are albuminous,—the red globules which contain the largest portion of that matter,—the white globules which seem to be more immediately concerned in nutrition,—the albumen of the serum,—and the fibrin, which is in the smallest quantity, and which differs from the albumen only in the peculiar (vital) attraction or aggregation among its particles; and which appears to exist in the living state partly, and, according to ANDRAL's observations, entirely, in the white globules above noticed. Until the relations of these different matters are better understood, we cannot explain how some of the most striking symptoms of that disease which seems to be the most directly produced by inadequate nourishment, viz., the Scurvy, are produced. But in that disease we now know that there usually is a great deficiency in the quantity of red globules, as well as either in the quantity or in the vital power of the fibrin; and we can now distinctly understand how it should happen that scurvy should shew itself, both when there is a long-continued deficiency of sufficient albuminous nourishment, and likewise when the nourishment taken is too exclusively albuminous;—most frequently, in this last case, when it is at the same time salted and hardened, and difficult of solution in the gastric juice, but, likewise, as repeated experience has shewn, when it is fresh and nutritious, but uniform.* In the first case (exemplified in several prisons of late years), there is a simple deficiency of azotised nourishment; in the other, there is a deficiency of the non-azotised matter which should protect this nourishment; the oxygen of the air therefore acts upon it, and the chief result seems to be, that the formation of the globules, apparently both of red and white globules, is prevented. Both cases are illustrated by what happens in BRIGHT's disease of the kidneys, where there is such a change in the vital action of these organs, that they throw off prematurely much of the albumen of the blood; the effect of which on the constitution of the blood is to diminish greatly all its azotised constituents, even although a full quantity of azotised food is taken; the specific gravity of the serum falling, and the proportion of the red globules to the other constituents of the blood becoming

* See BUDD on Scurvy, in the Library of Medicine.

as small as in the worst diseases of the stomach; while at the same time there is a tendency to extravasation, not indeed of the red globules as in scurvy or purpura, but of the serous part of the blood,—equally dependent as the extravasations in scurvy, on the condition of the blood itself.

But not only do we understand that there should be this great deficiency of the albuminous contents of the blood in scurvy, resulting after a time from the use of exclusively albuminous food, equally as from the denial of such food, or the continued morbid discharge of albumen from the blood, or the deficiency of digestive or assimilating power, as in chlorosis; but we understand, likewise, what appears at first sight paradoxical,—how the evils resulting from this state of the blood should be remedied by the use of food which is not albuminous, by succulent vegetables and vegetable acids. I do not say that we can understand exactly the efficacy of the small quantities of the vegetable acids in particular, which appear to be effectual in relieving the symptoms of scurvy; but we can distinctly perceive the principle, that, when a quantity of non-azotised matter is taken into the blood, the oxygen of the air will have less power to act injuriously on the albuminous constituents of the blood.

But although the distinction of the azotised and non-azotised ingesta, and the view taken of the chief offices of the two, enable us to understand much that was formerly obscure in regard to these points, yet it is not necessary, in acquiescing in this doctrine, to deny the possibility of the formation of albumen in the animal body. We may state other facts, occurring both in health and in disease, which are hardly consistent with the belief, either that no albuminous matter can be formed there, or that none of the albuminous matter taken into the body is applied immediately to the formation of excretions.

1. When we attend to the invigorating effect of pure air and of exercise on all vital action, and to the evidence we have of the increase of the red globules of the blood (the chief part of its albuminous constituents), and of the muscular texture throughout the body under their influence, it seems hardly possible to doubt, that the effect of the increased introduction of oxygen into the system is a real increase of the deposition of albuminous matter. Now, if there be no formation of albumen in the animal body, the increased introduction of oxygen is the increased application of a cause only of degradation or destruction of such matter; whereas, if albumen can be formed out of the non-azotised ingesta, as we have seen that there must be a considerable discharge of carbon and hydrogen, by help of the oxygen of the air, before the remaining elements can fall into the arrangement necessary for that purpose, we at once perceive that the effect of pure air and of muscular exertion must be, to increase the formation of that albuminous matter in the blood.

The effect of exercise in preventing or relieving the symptoms of Scurvy, ap-

pears to me peculiarly important in this inquiry. If we suppose that the immediate cause of the diminution of the albuminous matter in the blood, which takes place in that disease, is the action which the oxygen exerts on that matter,—in consequence usually of the small proportion of non-azotised matter which it finds in the blood,—and if the animal system has no power of forming albumen, we do not see how the increased introduction of oxygen should have any but an injurious effect; but if by means of it a part, even a small part, of the blood, consisting of amylaceous and oily matter, can be made to yield albumen, at the same time that it gives out carbonic acid and water, we can distinctly understand how the accession of scurvy should be retarded or prevented. And, in fact, we find that this effect is very generally observed, as the result of habitual and invigorating exercise.

It is stated by Sir E. Parry, that in Greenland the scurvy seldom makes its appearance among the natives until they confine themselves in their close huts for the winter, although the diet which they use when thus confined is the same as when they are moving about.

In our own country we have had various examples, on a large scale, of scurvy affecting prisoners long confined, although the diet on which they lived would not appear to have been materially different from that on which many of the lower ranks, particularly in Scotland, when at large, preserve their health, and are fit for much muscular exertion. Thus the diet of the prisoners at the Millbank Penitentiary in 1822, on which more than half of them became scorbutic (indeed three-fourths of those above three years confined), consisted of $1\frac{1}{2}$ lb. of brown bread daily, with one quart of soup, which soup had been made with from 2 to 3 oz. of the meat of ox-heads, with 3 oz. of garden stuffs, and was farther thickened with peas or barley; and at Coldbathfield Prison, about the same time, scurvy appeared pretty extensively within a few weeks after the diet had been reduced to $1\frac{1}{2}$ lb. of white bread, with 1 pint either of soup or gruel in the day, and $\frac{1}{2}$ lb. of beef on Sunday.* Comparing this diet with that of many labouring men in Scotland, consuming about $1\frac{1}{2}$ lb. of oatmeal, and *perhaps* 1 pint of milk daily, we can hardly doubt that the air and exercise of the latter exert an influence to improve the condition of the blood; whereas, upon the supposition that the oxygen of the air can give no help in forming albumen, that influence, in so far as the production of scurvy is concerned, should be only injurious.

2. All the phenomena of Scrofulous disease appear clearly to indicate that what we call the scrofulous diathesis, is necessarily connected with a deficiency in the nutritious or albuminous constituents of the blood; and we can now put that proposition in a definite and tangible form, in consequence of the important observation of Andral,—that in numerous trials made on the blood of persons

* See Holford's Second Vindication, &c. &c., pp. 4, 5, 10.

affected with tubercular disease, even in its earliest stage, he had always found the proportion of the red globules, in which the largest part of the albuminous matter is contained, less than the lowest proportion which he had ever found in healthy persons (less than 100 in the 1000 parts, the average proportion being 127). Now there is no proposition, in regard to the external causes of the scrofulous diathesis, which has been more anxiously investigated of late years, or, on the whole, more fully established than this, that it is, *cæteris paribus*, increased by atmospheric impurity and by sedentary habits, and diminished by pure air and exercise. Yet, if the animal frame cannot form albuminous matter, the only effect on the albuminous portion of the blood, of the increased introduction of oxygen which is implied in these circumstances, must be, to hasten the decomposition and expulsion of the albuminous matter absorbed from the primæ viæ. I do not state this fact, as affording more than a presumption against that opinion, because I am aware it may be said that, under the influence of fresh air and exercise, a larger quantity of albumen is taken into, or is absorbed from, the stomach and bowels, than in sedentary persons breathing impure air; but in so far as we can judge from the quantities taken into the body, I am pretty certain that the experience of medical men goes to prove that, when the quantities and kind of ingesta are *the same*, the beneficial effects of air and exercise in counteracting the scrofulous tendency,—*i. e.*, as I believe, in increasing the proportion of albuminous matter in the blood, may be distinctly perceived.

Indeed, independently of disease, I am strongly inclined to believe, that the nourishment of the animal body, and especially of the muscular textures, by a given quantity of ingesta, may be distinctly observed to be promoted by exercise, which is hardly conceivable on the supposition, that the only truly chemical changes which take place in the body are of the nature of oxidation, or slow combustion, and consequent excretion, in which the oxygen of the air is the chief agent.

3. The phenomena of Diabetes seem to me very adverse to the idea of the amylaceous matter taken into the system, being wholly inapplicable to the formation of albumen. In that disease, the digestion and appropriation of albuminous matter appear to go on even with unusual rapidity; and the urea which is contained in the urine, often in increased quantity in the early stage, and which is always easily obtained from it in full quantity immediately before death, shews that this matter is ultimately disposed of in the usual way in the animal economy; the amylaceous matter taken in must be the source of all the sugar which is formed in so great quantity, and which characterizes the disease; and it seems to be liable only to that kind of decomposition to which such matter is liable, by simply chemical affinities, at that temperature, and under the influence of water and oxygen; it is converted into sugar, and runs off by the kidneys, *i. e.*, it seems to be actuated by no vital affinity. Now, if all the starch taken into the

living body were useful, as this theory supposes, only by yielding to the simply chemical action of oxygen, and so giving off caloric, we do not see how these changes in diabetes should interfere with that office, or how they should involve so great derangement of the system, and particularly so much gradual wasting of all the textures. But if the starch taken into the system is liable to transformations resulting from vital affinities, and in which albumen is generated, then we can understand, that a disease in which starch seems to lose all tendency to vital action, and is rapidly thrown off, should be attended with this emaciation and debility.

4. When we attend to the phenomena of Lithiasis, *i. e.*, the morbid formation of uric acid, and the effects of different kinds of diet upon it, we meet with facts hardly to be reconciled to the idea of the albuminous ingesta being all destined for nutrition, and the non-azotised for combination with oxygen and excretion. It is well known, that Liebig pointed out that this diseased state depends on imperfect oxidation of the albuminous matter in the blood, which is destined to excretion (causing a formation of uric acid, when a fuller oxidation would produce urea and carbonic acid); and that he supposed all the albuminous matter which unites with oxygen in the blood, to be the product of absorption from the textures, the recently introduced albumen being, according to his theory, destined for nutrition only. Hence he argued, that a vegetable diet, increasing the quantity of non-azotised ingredients of the blood, with which the oxygen most readily unites, would leave less oxygen for the azotised or albuminous constituents, and aggravate the disease. But experience has shewn, particularly since the observations of Magendie were published, that the disease is more generally mitigated by a vegetable diet, under which, as it would appear, the whole quantity of azotised matter in the blood and in the urine is diminished, and the oxygen taken in is sufficient for its full oxidation. And the experiments of several authors have shewn, that the quantity of azotised matter thrown off by the kidneys increases greatly (may be nearly doubled) within a few hours after highly azotised food is taken. From which facts it would appear, that the azotised matter thrown off by the kidneys, is derived not merely from absorption of the textures, but likewise directly from the ingesta; and if so, the distinction of the azotised ingesta, destined only for nutrition, and the non-azotised, destined only for excretion, is not observed by nature; and it becomes extremely probable, that, as part of the albuminous ingesta are excreted, so a portion of fresh albuminous matter is formed in the blood, and applied, in the first instance, to the nutrition of textures.

IV. It is at all events certain, that Gelatin is formed in the living body, and its composition, as stated by Liebig, $C_{108}\ N_{18}\ H_{84}\ O_{40}$

or by Mulder, $C_{117}\ N_{18}\ H_{90}\ O_{45}$

compared with that of albumen, $C_{144}\ N_{18}\ H_{108}\ O_{42}$

seems evidently to denote that it is most probably formed from the elements of albumen, by a farther separation of carbon and hydrogen, aided by the agency of the oxygen of the air. LIEBIG seems to consider it as certain, that this separation must be from the elements of albumen, and, therefore, that gelatin can only be formed from albumen; but it is possible, also, that it may take place from the elements of starch with ammonia, oil being formed at the same time.

If we take the numbers given by MULDER as representing the composition of gelatin, this appears very distinctly. Thus,

	C	N	H	O
To starch,	120	...	100	100
Add ammonia,	...	6	18	...
	120	6	118	100
From this subtract,				
Elements of gelatin,	39	6	30	15
	81	...	88	85
And again, 5 equivalents of fat,	60	...	50	5
	21	...	38	80

which is exactly 21 equivalents of carbonic acid with 38 of water, excreted by the skin and lungs.

The "tritoxide of protein," lately so fully considered by MULDER, approaches so nearly in its properties to gelatin, that we may presume its formation will depend on nearly the same conditions; and accordingly we find, that it may be formed from albumen by the long-continued application of heat, air, and water; and that it is formed in large quantities in inflamed parts, where the stagnation of arterial blood (carrying oxygen) and the increased temperature plainly indicate that an increased application of oxygen is going on.

But as there is a remarkable discrepancy of statement as to the chemical relation of gelatin to the albuminous compounds, we must regard the precise nature of the change effected in this department of the animal economy as somewhat doubtful.

In thus attempting to trace the nature of the processes, wherever they may be carried on, by which carbon, nitrogen, hydrogen, and oxygen, uniting with other elements in smaller proportion, fall into the combinations which constitute the animal textures, and in attempting likewise to assign the province of the vital affinities in these processes, we must admit very material deficiency of information. We do not perceive, for example, how it should happen that the amylaceous matter, which forms the greater part of the ingesta of so many animals,

should hardly appear in their blood, even in that diseased state (diabetes) in which it passes off so copiously, in the form of sugar, by the kidneys. Neither is it easy to understand why the gelatin, formed probably in the course of the circulation, and deposited in so large quantities from the bloodvessels, should not appear in the blood. We are very imperfectly informed as to the origin, the use, or even the composition, of that animal matter, or rather congeries of animal matters, to which the name Extractive is applied. We are still in doubt as to the purposes served by the globules of the blood, both red and white, and the place and mode of their composition and decomposition.

But, admitting all these difficulties as to the details of the chemical changes, still these leading facts are ascertained :—that, in the cells of living vegetables, amylaceous, fatty, and albuminous compounds are formed,—and that, in the circulation through different parts of animal bodies, these compounds are selected and appropriated, and, in some instances, farther transformed, so that a farther formation of oily matter, and a new formation of gelatin, and probably of albuminous matter, takes place, applicable to the immediate nourishment of textures; that all these materials are formed ultimately from carbonic acid, water, and ammonia, existing in the atmosphere; that the carbon, originally fixed from the carbonic acid, is the most essential of all the ingredients, and the proportion of oxygen in all these organic matters, much less than in the inorganic compounds from which they are derived : that the affinities whereby the carbon is enabled to enter into these combinations with the other elements, existing in these organic compounds, to the exclusion of much oxygen, are peculiar to the state of life, and liable to variations by causes which do not affect dead matter; and that, in so far as the oxygen of the air is concerned in the formation of any of these compounds, it acts only by carrying off such portions of carbon and hydrogen, as enable the remainder of those elements to fall into certain new combinations with the others which are there present.

We may state another difficulty here, as leading directly to the next important question in vital chemistry, the rationale of the Excretions; viz., Why does the oxygen, which certainly attaches itself to the red globules in the lungs, not give evidence of its combining with the carbon in them, by giving them the dark colour, until it has passed along the arteries, and through the capillaries of the system, and entered the veins? This fact is noticed both by PROUT and LIEBIG. "The oxygen absorbed at the lungs," says Dr PROUT, "remains in some peculiar state of union with the blood (*query*, As oxygenated water, or some analogous compound ?) till the blood reaches the ultimate terminations of the arteries. In these minute tubes *the oxygen changes its mode of action ;* it combines with a portion of carbon, and is converted into carbonic acid."—(*Bridgewater Treatise*, p. 536.)

LIEBIG goes a step farther in explanation of the change of mode of action of the oxygen, when he says, "The globules of the blood serve to transport the oxy-

gen, which they give up in their passage through the capillary vessels. Here the current of oxygen *meets with the compounds produced by the transformation of the tissues*, and combines with their carbon to form carbonic acid, and with their hydrogen to form water."—(*Animal Chemistry*, p. 60.) But neither author has stated as clearly as I think may be done, on what principle it is that the oxygen changes its mode of action when it meets with these products of the transformation of the tissues; or, in simpler language, with the matters that have been absorbed from the living tissues. I believe the true reason to be, that this is an exemplification of a general principle of essential importance, which has been partially stated, but never, so far as I know, fully developed, viz., that *all vital affinities are of transient duration only;* and that those which actuate the matter of animal bodies especially, soon fail of efficacy, and at the temperature, and under the other conditions there present, give place to simply chemical affinities, which determine the formation of a very different set of compounds; therefore, that as long as the oxygen is passing along the arteries, and is in contact with albuminous matter, to which vital properties have been recently communicated, and which are actuated by vital affinities, it has little power to affect them; but when it meets with the same compounds in the substance of the textures, or already absorbed from them, *i. e.*, with albuminous or other animal matter, which, according to the expression often, but vaguely, used, has become *effete*, or has lost its vital properties, it can act on them in the living body in like manner as it does, at the same temperature, in the dead body.

But, in order to establish this point, it is necessary to enter on the second part of our inquiry into the chemical changes of animal bodies, *i. e.*, the peculiarities of the Excretions; *first*, of the greatest and most general of all the excretions from living bodies, the carbonic acid thrown off from the respiratory organs, both of animals and plants, of which Dr Prout says, that "the precise use of its constant evolution we know not,"—and *then*, of the other excretions from animal bodies. Until we have precise knowledge of the purpose which is served, and of the laws which are obeyed, by the matters which are continually expelled from living bodies, it is obvious that our notions in regard to vital affinities must be very unsatisfactory. In entering on this subject, I assume it as ascertained that all the matters, peculiar to the excretions from the living body, pre-exist in the blood, and are only eliminated from the blood at the organs where they appear; so that any chemical changes necessary for their formation, take place either in the cells of the textures, or in the circulating blood, or both, not in the glands which separate them, at least not externally to the vessels of those glands.

The first idea that must occur to every one who considers that large quantities of extraneous matter enter into every living body, different from those that can be traced in any of its textures, is, that the excretions from living bodies are simply those portions of the ingesta which are not applied to the maintenance of the or-

ganized structure. And that certain excretions are strictly of this character, seems to be fully ascertained, *e. g.*, the great excretion of oxygen from living vegetables, is merely separated from the carbon of the carbonic acid which enters them, when that carbon unites with the elements of water to form starch; and a part, at least, of the carbonic acid and of the water which are thrown off from a living animal, when it lives on sugar or starch, and forms oil or fat, or when it lives on albuminous compounds and forms gelatin, appears, from what was formerly stated, to be formed, by help of the oxygen of the air, from such portions of the carbon and hydrogen, of the starch or of the albumen, as are excluded when the new arrangement takes place, by which fat and gelatin are formed.

It is important to keep in mind, that, in regard to *all* the excretions, we have sufficient evidence of their being *partly* furnished in this way; *i. e.*, consisting of elements which have been taken into the body, but which are either redundant, or inapplicable to the nutrition of its textures; and that these are thrown off either alone, or combined only with a portion of the oxygen absorbed from the air, and the influence of which on the excretions will be considered afterwards. Thus it is certain, that part of the excretion from the bowels consists merely of unassimilated ingesta. It has been lately stated, with much probability, that certain matters in a putrescent state, absorbed into the circulation, find a natural vent in the mucous glands of the lower intestines.* When we consider that the bile is secreted chiefly from the venous blood of the vena portæ, and that this must necessarily be usually loaded with matters recently absorbed by the gastric and mesenteric veins, and not yet taken into the general circulation; and when we farther remember the small proportion of azote in the animal matter of bile, and the large quantity of this secretion in herbivorous animals, we can have no doubt that much of the matter (particularly the non-azotised matter) taken up by the veins, is brought to the liver only that it may be discharged thence in the form of choleic acid. We know likewise, that certain volatile matters, as alcohol or turpentine, however taken into the system, are excreted by the lungs, either unchanged or united (as in the case of phosphorus), with a certain portion of oxygen. And, in like manner, we have evidence, already stated, in regard to the secretion at the kidneys (although that evidence was not duly considered by LIEBIG), that a considerable part of it is frequently formed from matters recently absorbed into the blood from the primæ viæ, and which had never been applied to the nutrition of textures. As we know that the quantity of uric acid and urea, the most highly azotised of the animal compounds excreted, is much greater under the use of animal (*i. e.*, highly azotised food) than of vegetable, while the health and even the muscular strength

* See CARPENTER's Physiology, 3d edition, p. 685. This principle is probably of great importance in the pathology, both of hectic and typhoid fever, and of that form of dysentery which seems to result, as a specific inflammation, from certain putrescent miasmata.

may be equal; and that by the use of highly azotised animal food, the animal matter of the urine may be increased, according to CHOSSAT's observations, from 9·9 grains in the ounce to 17; and the proportion of urea voided may be even increased from 237 to 819; and, as we learn from the experiments of CHOSSAT, that a great part of this increase may take place within a few hours after animal food, rich in azote, is taken, we can have little doubt that a considerable part of that azotized food must have passed off by the kidneys without having been applied to the nutrition of any of the textures. And this appears to be confirmed by observations on that disease which arises from a morbid formation of uric acid in the system, because I think two facts may be regarded as nearly ascertained in regard to that state, viz., 1. That it depends essentially on imperfect oxidation of the azotised matters contained in the blood, and destined to excretion;* and, 2. That it is most generally and effectually diminished by a vegetable diet, lessening the quantity of azotised matter taken into the body; whereas, if all the azotised matter destined to excretion had been the production of absorption in the body itself, the introduction of much non-azotised matter, with which the oxygen of the air certainly combines in the circulation, would have left less oxygen to unite with that effete azotised matter, and would have determined, therefore, a greater production of the imperfectly oxidised uric acid, as proportioned to the urea.†

These facts seem sufficiently to illustrate and justify the common opinion, that the excretions are furnished, in part, by such portions of the ingesta as are either inapplicable to nutrition or redundant; and which are, therefore, either

* This is shewn thus—

	C	N	H	O
Uric acid	100	40	40	60
Add water	...	...	40	40
„ oxygen,	...	...	...	60
	100	40	80	160
Subtract urea,	40	40	80	40
	60	...	...	120 = 60 CO_2 Carbonic acid.

† LIEBIG, taking for granted that it is the non-azotised portion of the ingesta only, that is united with oxygen from the air in the course of the circulation, thought the use of vegetable food improper in this state of the body, as absorbing the oxygen, and causing, therefore, imperfect oxidation of the azotised matter absorbed from the textures, and about to form urea and uric acid. But the observations of MAGENDIE and others, shewing that both in health and disease the proportion of uric acid formed is generally less under a vegetable diet than an animal, particularly when taken in connection with the facts stated above as to urea, must be regarded as proving, that the idea of non-azotised food having that exclusive tendency to unite immediately with oxygen in the blood, must be erroneous.—See *Carpenter's Physiology*, § 849, 850.

excluded from the new combinations which are formed in a living body, or rejected from the selections which are there made.

Now, if we consider it as ascertained, that a part of all the aliments taken into a living animal body, combines immediately with the oxygen of the air, in the blood, and is thrown off by the excretions in the form of water, carbonic acid, and ammonia,—or in forms which tend towards, and quickly resolve themselves into, these compounds,—we see a distinct confirmation of what was formerly stated, as to the nature of vital affinity, viz., that it does not, properly speaking, supersede ordinary chemical affinities, but is merely superadded to them; so that chemical compounds, taken into animal bodies, are subjected to these attractions as well as others, and are divided between the substances thus acting upon them, in proportions varying probably, as in other cases, according to the strength of the affinities and the quantities of matter exerting them. This, indeed, appears sufficiently demonstrated by the effect of exercise (already considered) on the excretions by the skin and lungs, on the one hand, and on the deposition of fat or of albuminous compounds, on the other; we know, that, as the quantity of carbonic acid and water thrown off are increased by that cause, the quantity of fat deposited from the blood is diminished,—implying that, by the increased quantity of oxygen presented to them by the blood, portions of the carbon and hydrogen of the ingesta, which would otherwise have been subjected to the vital affinity which forms fat, have yielded to the simply chemical affinity which disposes them to unite with oxygen and pass off; and again, it is at least highly probable, that, under this increased supply of oxygen, increasing, by a simply chemical attraction, the proportion of carbon and hydrogen which escape from the ingesta, the effect of the vital affinity by which the remaining elements of the ingesta combine to form albuminous matter, is likewise increased.

But we have next to consider the evidence for the existence, and the object and importance of another and totally distinct source, long believed to contribute to the formation of the excretions, viz., matter which has formed part of the textures of the living body, and been re-absorbed from them, with the intention of being thrown out of the body; *i. e.*, the dependence of excretion on what Dr PROUT calls "destructive assimilation."

The mixture of this matter with the blood appears to be necessary for all the changes there, from which the different excreted fluids result; or, it may be supposed not merely to escape itself, but to act as a ferment, promoting these changes, and thereby determining the entrance into these combinations, and the expulsion from the body, of the portions of the ingesta which are not required for nutrition.

The term *effete* matter has been very generally employed in discussions on this subject; but it does not appear to me, that any very definite idea has been annexed to the term, nor that any principle has been pointed out to explain how

animal matter becomes effete,—why the absorption of matter once deposited in the textures should be a necessary concomitant of animal life,—or why the elements composing these textures should enter into new combinations, and then should require to be expelled from the body. But I am persuaded it will appear, on examining the subject, that the principle formerly stated, of the *transient existence of vital affinities* in every portion of matter which becomes endowed with them, is both supported by sufficient evidence, and adequate to the explanation of these phenomena.

The leading facts on which this conclusion may be rested are the following:—

1. We know that a continual process of absorption and change of materials is always going on in every living animal texture, and is, in fact, the cause why a continual act of nutrition (the most characteristic of all the functions of animals) is essential, not only during growth, but even in the decline of the body, to the maintenance of its structure and properties.

2. We know that, simultaneous with this absorption, there is a continual process of excretion going on from every living animal, and that, by these excretions, a quantity of all the elements constituting the animal textures is continually thrown off; and farther, it appears to be indicated, although I cannot say fully established by Liebig, that the sum of the chemical elements thrown off by the different excretions sufficiently accounts for (the presence of oxygen and water being kept in mind), not merely the *part* of the blood which is not applied to the nourishment of the textures, but the whole of the blood.*

3. We know that the excretions, at least that some of them, not only continue but increase, particularly under any increased muscular exertion, and that their nature remains the same, in an animal deprived of aliment, and in a state of rapid emaciation, as in one that is fully supplied with aliment, and perfectly nourished. "In a starving man, who is in any way compelled to undergo severe and continued exertion," says Liebig, "more urea is excreted than in the most highly fed individual, if at rest. In fevers, and during rapid emaciation, according to Prout, the urine contains more urea than in health."†

While these facts prove incontestably that a great part of the matter thrown off from every living body must be the product of absorption from the body itself, let us next consider the information that we have, as to the change which is wrought upon the absorbed materials before they are expelled from the body.

1. The most leading fact in this part of the subject is, that, in the natural state, *none of the organic compounds which exist in the textures, appear in any of the*

* See Animal Chemistry, p. 136 and 152. This conclusion, however, is not to be regarded as established, various fallacies being connected with it. In fact, it seems to me only certain that the carbon and nitrogen are in the same proportions in the excretions as in the blood.

† Animal Chemistry, p. 139.

excretions, although it can only be through the excretions that they disappear from the body, and although the earthy or saline matters absorbed from the textures are there found. The animal compounds existing in the textures must therefore have undergone a great chemical change, in the process by which they are removed from their place in the living body, and finally expelled from it; and this notwithstanding that they are placed in circumstances exactly similar to those, in which their previous original separation and deposition from the blood in the minute capillaries took place.

2. The substances into which these animal compounds (with or without additions derived directly from the primæ viæ) have resolved themselves almost entirely before they are thrown off in the excretions, must be, the water which is the basis of all, the carbonic acid thrown off by the lungs and skin, the choleic acid thrown off by the liver, and the urea and uric acid thrown off by the kidneys. All these last we know to be formed in the course of the circulation, not in the organs by which they are separated from the blood; and all possess these essential peculiarities, distinguishing them from the compounds forming the textures; *first*, that they are crystallizable, *i. e.*, the elements composing them are so arranged as to be capable of assuming the definite forms peculiar to inorganic matter; and *secondly*, that they are poisonous to the living body when they are allowed to accumulate in the blood, and, therefore, that their continual expulsion is essential to life.

3. When we farther examine these compounds, into which the animal textures have resolved themselves before they are expelled from the body, we find that they are substantially the same as those, into which these textures are ultimately converted after death, by help of union with oxygen, when in contact with air and water, and at a certain temperature,—viz., water, carbonic acid, and ammonia, the small quantities of sulphur and phosphorus contained in the animal textures, combining likewise with oxygen so as to form sulphuric and phosphoric acids before they are expelled.

	C	N	H	O	
Thus Urea consists of	100	100	200	100	
Add water,	...	...	100	100	
	100	100	300	200	= Carbonic acid and ammonia.
Again, choleic acid consists of	76	2	60	2	
Subtract urea	2	2	4	2	
	74	...	56	20	Adding oxygen freely,
	...	...	...	184	
We have,	74	...	56	204	= 74 CO_2 + 56 HO carbonic

acid and water.

Thus the general fact seems established, that the excretions from the living body are only an intermediate stage between the organic compounds, forming the animal textures, and the inorganic chemical compounds into which these are ultimately resolved after death; and that in the same living body, and in the same parts of it, at the same temperature, and when in contact with the same substances, the same chemical elements, carbon, nitrogen, hydrogen, and oxygen, are continually acting on one another so as to form two distinct sets of compounds; the one set peculiar to living bodies, always attaching to them certain saline and earthy matters, sulphur or phosphorus, and always taking the form of cells or fibres, never of crystals,—and building up the organised frame; the other set rejecting those adventitious matters, tending always to the crystalline forms, and to the same mode of combination of the elements as takes place, under the same temperature, where no living structures exist,—and which are always expelled from the organised frame. These are facts of such obvious importance, so generally observed and characteristic, that the physiologist cannot decline to take cognizance of them, and arrange them together, and have some general expression for them. It does not appear possible to express these facts otherwise than by saying, that the particles of these elements taken into living bodies, are under the influence of different chemical laws at different times; which is exactly what we mean by saying, that they are first actuated by vital affinities (called vital because they are seen only in living structures, and in connection with the indications of life), by which the organised structure is gradually formed, and afterwards by simply chemical affinities by which it is gradually worn down; and that both are in continual operation during life. And thus it appears that the chemical change, which always attends the absorption, and discharge by the excretions, of all parts of a living body, is simply this,—that they lose their vital properties, and become liable to the same affinities among themselves, and the same action with the oxygen brought to them by the blood, as prevail in the dead state.

This inference as to the loss of vital properties, has been stated by several authors of late years, in regard to those portions of the living solids which perform distinctly vital actions in a visible or tangible form, as the portions of muscular fibre or nervous matter, which are employed in vital motions and sensations; but as the facts from which we draw the inference are equally true of bones and membranes, and other animal solids, unconcerned in any such vital actions, it seems to me necessary to extend the inference to all those portions of matter which exhibit in a living body the vital affinities, as well as to those which take on any kind of vital movement, or are concerned in any nervous actions.

That oxygen must be the main agent in effecting the changes of these animal compounds, which precede their expulsion in the excretions, is sufficiently proved by observing, *first*, that it is uniformly and necessarily applied to them when these changes are going on; *secondly*, that the compounds into which the animal

matters are converted before they are excreted, contain a much larger proportion of oxygen than those compounds themselves; and, *thirdly*, that it is also necessarily applied to all dead animal matter when the decomposition, leading to the same ultimate results, takes place in it.

It is true that the Bile does not contain a larger proportion of oxygen than albumen, but it contains a larger proportion than any kind of oil or fat, from which it appears certain that it is partly formed; and, farther, we have perfectly good evidence, very well stated by LIEBIG, that by far the greater part of the bile in all animals, and nearly the whole in the carnivora, is re-absorbed into the blood, and exposed gradually to the action of oxygen on it above indicated, and therefore that the secretion of the liver, so far as it is destined to excretion, resolves itself chiefly into the excretion of carbonic acid and water by the skin and lungs, and partially also into that of urea and uric acid by the kidneys; which arrangement, we have reason to believe, is designed with a view to the maintenance of animal heat, to be considered afterwards.

It may here be a question, whether the simply chemical attraction of the oxygen, carried to the extremities of the vessels in the blood, is the cause, or part of the cause, of the act of absorption, antagonizing the strictly vital attraction by which the elements of nutrition are brought into the cells of the textures. But the power exercised by the excretory glands themselves appears manifestly to be merely that of selection and attraction of the material destined to pass out by them, by an agency of cells quite analogous to that by which the cells of the textures appropriate their own nourishment; and by this simple and beautiful principle, of certain cells, or the cells in a certain part of the structure, exerting a peculiar attraction for certain matters only, existing in the compound fluid presented to them, nature has provided both for the nutrition and growth of all the textures, and for the expulsion of such matters as must be evolved from the blood, and have not such a property of volatility as might enable them to pass off by the skin and lungs.

It may be objected to the statement now made as to the respective provinces of vital and simply chemical affinities, that vegetable and animal substances removed from the living structures which formed them, are often of long and nearly indefinite duration; but it would be an error to infer from this fact, that the affinities which led to their formation act as long as they endure; we can only infer that the conditions, under which other chemical affinities act on such compounds, are not present; and the general property of the inertia of matter prevents their changing the condition into which they have been once brought, just as the same substance reduced to the state of charcoal may remain long unaltered, although in contact with oxygen, and liable to an affinity with that gas, which, under a slight variation of circumstances, would convert it into car-

bonic acid. "There exists," says LIEBIG, "in every compound a statical momentum (*moment statique*) of the attractive powers which combine the elements; the inertia of the elementary atoms, or their disposition to persist in the same state, or in the same place, where they actually exist, acts there as a special force. If the atoms of sugar were held together by as strong a force as the elements of sulphate of potass, they would suffer as little disturbance as these, from the presence of a ferment or a putrescent body. But this is not the case. The elements of all organic compounds which are capable of undergoing transformations preserve their condition only in virtue of the *inertia*, which is one of their properties."*

Again, it has been reasonably objected to the doctrine of the nutrition and growth of animals being due to an affinity between their textures and the ingesta taken into them, which ceases when these ingesta lose their vitality, that these aliments are very generally in a dead state before they are submitted to the organs of digestion.† But I apprehend the proper answer to this to be, that,—so far as the chemical phenomena of life are concerned, the death of an entire living structure is quite distinct from the death of any one of its component parts. The whole of a living structure dies when its nutrition, the most essential of its functions, is brought to a stand by the failure of circulation; but the organic compounds, formed, as I believe, by vital affinities in that structure, remain for very various periods of time unaltered, or are preserved, as LIEBIG expresses it, by the inertia of matter, from forming those inorganic compounds to which they are ultimately destined; and as long as they remain *fresh*, or, although undergoing decomposition, have not yet reverted to those inorganic compounds, they seem to be still capable of being acted on by the vital affinities of animals. But, when the simply chemical affinities have really resumed their power, when a part of the body has undergone a certain degree of putrefaction,—when the carbon of these compounds has passed into the state of carbonic acid,—or even when this and the other elements have combined so as to form the excretions, which are steps in the process by which they revert to carbonic acid, water, and ammonia,—they are no longer capable of being applied to the nutrition of animal bodies, until they have been again subjected to the influence of vegetable life. The fact of their falling into the combinations which form the excretions, in the act of absorption from the living textures, must be regarded as proof that they have lost their own living properties, and can no longer form part of a living texture, although still within a living structure. This death of the individual molecules forming the living textures, I take to be the counterpart of the continued nutrition of those textures during life, as a general fact in the history of

* "Sur les Phenomenes de la Fermentation," &c. Annales de Chimie, t. lxxi., p. 19, 193.

† See the Review of PROUT'S 4th edition, in British and Foreign Review.

living animals. It is by thus losing their vitality that these molecules become liable to the interstitial absorption (of HUNTER); and their places are taken by fresh molecules by virtue of the vital attraction which constitutes nutrition.

It appears certain also, that the healthy exercise of the vital functions of any texture (although within certain limits it strengthens all the vital properties, and augments the living structure, apparently by attracting an increased flow of blood) determines the more speedy death of the molecules composing it, and the more rapid change of its particles by absorption. This may be expressed by saying, that this mode of vital action, as well as all muscular and nervous action, is subject to the general law of alternate increase and diminution. Hence the increase of absorption, and therefore of the excretions from exercise, even when all ingesta have ceased. And hence, also, if the vital act of nutrition in any texture is morbidly excited, as happens in every case of inflammation tending to the formation of plastic lymph, we have subsequently an increased loss of vitality in the molecules of that part; and therefore, either the formation of purulent matter destined to excretion, or the increased absorption of the newly formed or effused lymph, or the ulcerative absorption of the solids previously existing, or sloughing, or gangrene,—all well-known results of the inflammation, but which have not been duly regarded as all implying more or less partial *loss of vitality*, and therefore dependent on the same principle; and which experience shews to be linked together and even to graduate into one another.

In like manner the progressive absorption of HUNTER is probably to be ascribed to the influence of pressure, injuring and permanently destroying the vitality of parts not intended nor fitted to undergo pressure, and thereby preparing them for absorption and for the action of oxygen.

It is hardly necessary to add to this statement, after the researches of DULONG and DESPRETZ, of DUMAS and of LIEBIG, that the combination of oxygen with the other constituents of the excretions, and particularly with the carbon and hydrogen, is (as has always been maintained by most physiologists in this country) the true cause of Animal Heat; and it cannot be doubted that one of the uses of the aliments, especially the non-azotised aliments, continually taken into the body, is merely to enter into this combination, and fulfil this purpose. But there is one principle on this subject, not so generally recognised, but which the observations of LIEBIG, and likewise of SCHERER, of PETTENHOFFER, and of BOUCHARDAT and SANDRAS,* seem to make nearly certain, viz., that a principal use of the secretion of the Liver (*i. e.* of the animal matter there secreted) is, to serve as a reservoir for the most easily combustible matter which is taken into the primæ viæ; so that,—just as the chyme of the stomach and intestines furnish a pretty constant supply of nourishment from occasional supplies of aliment,—

* See PAGET's Report in FORBES's Journal, April 1846, pp. 561 and 562.

so the Bile from the liver, likewise reabsorbed as it passes down the primæ viæ, furnishes to the blood a pretty constant supply of matter fit for calorific combination with oxygen, out of the occasional ingesta.

The proofs of this proposition, and its importance, appear from the following facts, ascertained by these authors. 1. That by far the greater part of the amylaceous matter taken into the stomach, is converted into soluble matter (dextrine and sugar) in the primæ viæ, and these must necessarily be absorbed by the veins, and of course carried to the vena portæ and liver. From thence a part of this matter, no doubt, will pass immediately by the venæ cavæ hepaticæ to the right side of the heart and lungs, and come immediately into contact with the oxygen; but a part, meeting a portion of effete animal matter in the venous blood will aid in the formation of bile in this way:

	C	N	H	O
4 equivalents of starch	48	...	40	40
Add 1 of ammonia	...	1	3	...
	48	1	43	40
Subtract elements of choleic acid	38	1	33	11
	10	...	10	29

requiring only one part of oxygen to pass into 10 CO_2 + 10 HO, carbonic acid and water; which accounts for the great quantity of bile secreted by herbivorous animals; and accounts likewise for the secretion of bile being chiefly from venous blood, inasmuch as very little oxygen is required for its formation, and its chief pabulum has been recently absorbed by the veins. In so far as bile is formed from fat, it must be by help of more oxygen, and, therefore, probably from arterial blood.

2. That of the bile formed and discharged into the intestines, the greater part, even in the herbivora, and almost the whole in the carnivora, is reabsorbed into the blood, and decomposed in the process, the pure bile appearing distinctly in the fæces almost exclusively in the case either of diarrhœa, or of the operation of cathartics. When to these facts we add these considerations, that biliary matter retained in the blood, as in one form of jaundice, acts as a poison, and that it cannot be of use in the nutrition of the textures, which is provided for by the albuminous contents of the blood, we can hardly doubt that it is reabsorbed into the blood, only that it (or its elements) may unite with oxygen, and be thrown off as carbonic acid and water, with a little urea; and therefore, that the liver is an appendage to the digestive organs, destined for the proper disposal of the calorific, rather than the nutritious portions of the food, and for the necessary separation of these two; and that the circulation of the matter destined to this

ultimate object, through the liver, answers the important purpose of equalizing the quantity of matter in the blood, which is always ready for this calorific union with oxygen.

This doctrine, as to the chief use of the animal matter of the bile, appears to correspond perfectly with several known and important facts. When the quantity of bile thrown off by the liver, and discharged by the bowels, is decidedly greater than usual, the animal heat is remarkably depressed, as in cholera, apparently because the quantity reabsorbed and applied to the evolution of heat, is diminished. In herbivorous animals, the quantity of bile discharged from the bowels is much greater than in the carnivorous, because the quantity of amylaceous matter which they consume is so much greater, that a much larger quantity is secreted, and if all reabsorbed into the blood, it would cause a morbid increase of heat. Again, in warm climates and seasons, the formation of bile is apparently stimulated, the liver is excited to increased action, and there is such an increase of the discharge by the bowels, as serves to lessen the quantity of combustible matter in the blood, and keep down the temperature of the body; but then this increased stimulation of the liver renders it more liable to various forms of disease.

When we say that oxygen, acting on the redundant, on the non-azotised, and on the effete matters with which it meets in the blood, is the main agent in forming the excretions, and causing the waste of the body, we use language which is, to a certain degree, ambiguous. It seems to me that the oxygen is probably capable of acting on all the matters in the blood for which there is no strong vital affinity in the body; and that the action of the oxygen on the matters which are ready to be, or have been, absorbed from the textures, is rather the consequence, than the cause, of their having lost their vital properties, and thereby come under the dominion of ordinary chemical affinities. The oxygen is, no doubt, the agent by which the gradual extenuation of the body, in death by famine, or by mány lingering diseases, is effected, but this agency of the oxygen is in itself salutary, and even necessary to life; the real cause of death is, that cause which prevents the loss of substance effected by the oxygen from being immediately repaired, *i. e.*, it is the deficiency of nourishment, to take the place of those portions of the textures which have lost their vital properties, and therefore come under the dominion of the oxygen.

This seems to be confirmed by the fact which appears to have been fully ascertained by CHOSSAT, that the rate of waste, *i. e.*, the rapidity of absorption of the textures of the body, is greatest shortly before death, *i. e.*, when the supply of the oxygen must be diminished, rather than increased, from the state of the circulation and respiration,—but when the vital powers, and especially the vital affinities, are losing their power, and the supply of nourishing matter has ceased. This fact alone seems sufficient to shew that the absorption, which is constantly

going on in the textures while life continues, is due to the partial loss of vital power of these textures themselves, and is the cause, rather than the consequence, of the agency of oxygen upon them.

When we consider, farther, how exactly this is in conformity with the general fact, that all other kinds of vital action are *essentially temporary*,—that all nervous actions, and all muscular contractions, necessarily alternate with periods of repose,—I think we can have no difficulty in acquiescing in the general law of all Vital Affinities, at least so far as animals are concerned, which explains at once the necessity of constant nutrition of all animal bodies (even when their weight is stationary or declining), the principle of interstitial absorption, the use of respiration, the maintenance of animal heat, and the necessity and nature of the excretions; viz., that as the perpetuation of each species is provided for only by the successive life and death of numberless individuals, so *the life of each individual is sustained only by the successive life and death of all the portions of matter of which its body is composed;* and that each portion, as it dies, falls under the power of the oxygen absorbed from the atmosphere, as it would do in the dead body, and enters into new combinations which are injurious to the living system, but pass off by the excretions; gradually reverting to those inorganic compounds, from which the power of vegetable life only can again raise them to the condition of organized and living matter.

The general conclusions regarding Vital Affinity, which seem to me to be warranted by this review of the subject, and to be sufficiently established to be stated as principles in Physiology, are the following :—

1. That it is by a power peculiar to the state of life, and equally vital as the irritability of muscles, but varying in the different parts of each organized structure, that the solids, and especially the cells of organized matter, attract, select, consolidate, and arrange in their substance, and within their cavities, certain substances, usually compound, which are brought into contact with them, and reject or exclude others.

2. That in the cells of organized matter, during the living state, and apparently by an influence of these cells analogous to that chemical influence to which the term Catalysis has been applied, analogous also to fermentation, certain definite transformations of chemical elements take place, which are equally peculiar to the state of life; which transformations, at least in animals, appear to be effected more in the cells or corpuscles which float in the fluids, than in those which compose the solid part of the structure.

3. That although we have proof that the origin of all the organized beings now seen on the earth's surface has been of recent date, in comparison with the earth itself, we see these powers, thus exercised, continually transmitted to successive sets of cells in each individual, and to successive generations of individuals, with-

out being able to remount to the origin of this kind of action in this, as in others of the sciences lately called palætiological.

4. That the first essential condition necessary for the development of all organized life, is that vital affinity by which, under the influence of light, the cells of vegetables appropriate and decompose the carbonic acid of the atmosphere, fix the carbon, and attach to it the elements of water, so as to form amylaceous matter.

5. That the ulterior changes, effected within organized structures, by which oily, albuminous, gelatinous, and perhaps extractive compounds, are formed and assimilated to the living textures, appear to belong to certain definite vital affinities of the carbon, originally fixed from the air, and which is the basis of all organized substances, not only for the elements of water, but for hydrogen, for azote, for sulphur, phosphorus, and various salts; that most or all these ulterior changes are effected both in vegetables and animals; and that the oxygen taken in by the organs of respiration, although it may be necessary to the play of all the different affinities in living bodies, appears hardly to enter, if it enter at all, into the constitution of any of the compounds thus formed and applied to the nourishment of the textures.

6. That these compounds, in order that they may be applied to this purpose, must be moved within living bodies, and applied, in the fluid form, to the textures which they are to nourish, although in various instances, both in vegetable and animal life, they have themselves the solid form; and that the requisite fluidity is given by various contrivances, chiefly seen in the *primæ viæ* of animals,—by mechanical attrition, by incipient decomposition of the materials employed, but especially by a simply chemical solution of these,—for which purpose certain parts of living structures are endowed with a vital power of separating acids, and others of separating alkalies out of the compound fluids pervading them, and thus preparing solvents for those solids.

7. That the vital affinities do not, strictly speaking, supersede ordinary chemical affinities in the living animal body, but are superadded to them, so that the ingesta, as they come under their influence, are divided between the combinations to which those different kinds of affinity dispose them, and particularly are partly under the influence of the substances exerting vital affinities, and partly of the oxygen of the air, brought to them by the arterial blood; and that as these ingesta often contain large quantities of matter, especially of non-azotised matter, either inapplicable to the formation of the animal compounds, or redundant, these portions, fall immediately under the influence of the oxygen, and form one source of the excretions from the animal body.

8. That the vital affinities, like all living properties, are liable to an influence of *place* and of *time*, which is not seen in the inorganic world, but is an essential attribute of the organized Creation, which has been superadded, in later times, to the original arrangements of the universe. They are acquired by por-

tions of matter which are brought to particular points in previously existing organized structures; they are vigorous for a time, and are then lost. In all the compounds constituting the animal textures, these affinities become gradually enfeebled, whereby the elements constituting these textures become liable to absorption into the blood, to changes in their arrangements, chiefly effected by the oxygen of the air, to combinations with the redundant matters above noticed, and to the formation of other compounds in the blood, which are either the same as, or rapidly tend to, the combinations with oxygen to which animal matter is liable in the dead state; which are, therefore, properly speaking, due to simply chemical affinities, and therefore crystallizable, like other inorganic compounds, and are noxious to the animal economy. This is another source of the excretions, for the separation of which appropriate organs are furnished, capable by their vital power of absorbing and abstracting them from the blood.

9. That the simply chemical power thus exerted by the oxygen, taken in by respiration, over the redundant (especially non-azotised) matter in the blood, and the *effete* matter of the textures, is the source of Animal Heat.

10. That there is thus effected during the life of animals, but in consequence of the failure of their vital affinities, and restoration of the simply chemical relations of their component elements, a change equivalent to the slow combustion of the organized matter, which had been first prepared by the vital affinities of vegetables; and that the carbon, hydrogen, and other elements employed in the formation of that matter, are thus continually resuming that condition, from which the power of vegetable life is continually abstracting them again, to communicate to them a set of properties at variance with those which they permanently possess; and apply them to a succession of organized beings which can only terminate, as at no very distant period of time it must have originated, by an arbitrary act of Divine power.

The gradual change both in vegetable and animal structures which results from age,—the increase of the proportion of earthy and saline matter, and diminution of the proportion of strictly organic matter,—must be regarded as indicating a peculiarity of the vital affinities equally an ultimate fact as their limited duration in every portion of a living body. And the modification to which these affinities, as well as all other strictly vital powers, are liable in animals, from certain actions of the nervous system, must likewise be regarded as an ultimate fact, quite distinct from any principles that have been ascertained in regard to the nature of the vital affinities themselves.

On reviewing the statements and reasonings which I have laid before the Society on the subject of Vital Affinity, although I may have committed errors in the details, I cannot accuse myself of having occupied their time, either with a vague and useless speculation, or with a verbal dispute.

That there is *something* in the history of all living bodies which is *peculiar* to

them, at variance with the laws that regulate the changes of inorganic matter, and requiring to be investigated by a separate induction of facts, must be admitted by all; and is indeed the only reason we can give for treating Physiology, and the branches of knowledge dependent on it, as a separate science; and this being so, it belongs to the very elements of the science to determine what are the portions of the history of living bodies which come under this category.

I have always held in high respect the aphorism of HEBERDEN, which Dr GREGORY used to recommend to the special attention of his pupils, that the great desideratum in medical science is the detection of the Vital Principle, by which all that goes on in the living body is regulated and governed; but I have always thought likewise, that the object of this investigation is rightly limited by Dr PROUT, when he says that we should inquire, "not what the vital principle or vital power *is*, but what it *does*." In fact, in all the sciences, we can acknowledge only one principle and one Power, as the origin of all the phenomena that we investigate; and when we use these terms in reference to living beings,—when we say that we inquire how the vital principle acts,—we use the term only as a convenient and simple expression for an investigation of the laws according to which the Divine power acts, in regulating the changes which are continually taking place in the last and noblest of the works of creation, and which differ from the changes that we see around us in other departments of nature.

This precise and definite object of all physiological researches—the determination of the laws that are *peculiar* to the science—has always attracted the attention of physiologists, but has not always been placed in the proper point of view; and the common error in this, as in other sciences, has been, to regard the laws of nature as simpler than they really are, and to stretch a principle, ascertained as to one set of phenomena, in the hope that it would be found sufficient to embrace many more. Thus it was easily observed that the phenomena of sensation and thought, and the visible motions in animals, were quite peculiar to them; and when it was ascertained that the first of these, and that a large portion of the latter (viz., all voluntary motions), depend on the living state of the nervous system, it was hastily concluded that all the phenomena peculiar to animal bodies, depend on their Nervous System. This is illustrated by the title of one of the chapters in GREGORY'S "Conspectus." "De solido vivo, seu genere nervoso," as if there were no living property in any of the animal solids but what is given to them by the nervous system; or, by the explicit declaration of CULLEN, that he considered the vital principle as "lodged in the nervous system."

The progress of the science has, I think, distinctly shewn that these ideas, as to the parts of the animal economy in which the peculiar laws of vitality operate, were limited and erroneous; although physiologists (trained in the schools of medicine where the authority of these and other teachers, adopting similar doctrines, has been held in just veneration) have been generally reluctant to admit the error.

I have endeavoured, in papers laid at different times before this Society, to

limit and define our notions of the powers exercised by the Nervous System, in producing the phenomena of the life of animals, maintaining on that subject the different parts of one general and fundamental proposition; viz., that there is no good evidence, and that in the absence of such evidence it is unphilosophical to assume, that any changes in the nervous system are essentially concerned in producing any phenomena in the healthy state of the system, except those in which *some mental act is necessarily involved;* but that all the powers which are exercised, in the natural and healthy state, by the nervous system, in a living body, are those by which it fulfils its destined office as the seat, and the instrument, of mental acts,—of Sensation, Thought, and Instinctive or Voluntary effort; and that the nature of these powers, and the uses or intention of the different parts, and of all the arrangements of the nervous system, if judged of simply in reference to these, the specific objects of its creation, are tolerably well ascertained; vindicating, at the same time, the doctrine of HALLER, in regard to the separate vital property of Irritability or Contractility in muscles, and its different modes of connection with the nervous system.

I likewise stated, on a former occasion, to this Society the evidence of another fundamental principle in physiology—of the existence and the chief agencies of a power exercised by living bodies, and peculiar to their living state—which is capable of producing motion, or of influencing motion otherwise produced, but which acts in the way of Attraction and Repulsion; and is, therefore, quite distinct from that living power of animal solids, acting in the way of contraction and impulse, which is well understood; and to which, since the time of HALLER, the name of Irritability, or the more general term Contractility, has been applied.

Although both these principles have been strongly contested, I have had the satisfaction of seeing them adopted, and their importance acknowledged, by most of those who have prosecuted the science of Physiology in this country of late years, with the greatest diligence and success. I have now laid before the Society the general grounds of a third opinion, which I hold to be of equal rank in physiology; viz., that there are laws, peculiar to living bodies, acting to a limited extent only, and already in a considerable degree ascertained, which alter and control the ordinary chemical Affinities of the matter composing those bodies, as distinctly as the laws of muscular contraction, or of vital attractions and repulsions, modify the effects of the ordinary mechanical properties of matter within them. And if this doctrine shall, as I confidently expect, be equally admitted to be correct, then, although laying claim to no credit as a discoverer, I hope I may be allowed the satisfaction of reflecting, that I have contributed somewhat towards fixing the foundations of the noble science of Physiology; and establishing those principles in that science, to which continual reference must necessarily be made, in any speculations to which we can apply the epithet scientific, in regard either to the nature of diseases or the operation of remedies.

XXII.—*An Attempt to Elucidate and Apply the Principles of Goniometry, as published by* Mr WARREN *in his Treatise on the Square Roots of Negative Quantities.* By the Right Reverend Bishop TERROT.

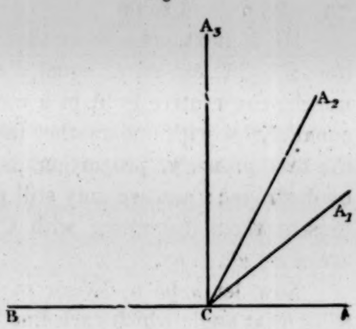
(Read 18th January 1847.)

1. The symbol $\sqrt{-1}$ is called an *impossible* or *imaginary* quantity, because, in analogy with the received laws of algebraic symbolism, it must mean such a quantity as, being multiplied into itself, gives for a product -1. Assuming, then, that every quantity must be either *plus* or *minus*, it follows that the square of every real quantity must be *plus*; and hence $\sqrt{-1}$, which gives its square *minus*, is called an imaginary or impossible quantity.

If, however, we consider the most simple application of algebra to geometry, we shall perceive that the assumption that every line must be considered and symbolized as either + or −, is inconsistent with fact. In algebraic geometry, $+a$ or $+1\times a$ symbolizes a line whose numerical length is a, drawn in some given direction; while $-a$ or $-1\times a$, symbolizes a line of the same length, drawn from the same extremity in the same straight line, but in a directly opposite direction. To say, then, that all lines must be either + or −, is as much as to say that all lines drawn from the common extremity must be drawn in this one assumed line; and that it is impossible any line should be drawn making an angle with it. But it is evident that an infinite number of such inclined lines may be drawn, and none of them can have $+1$ or -1 as a factor, in accordance with the definition just given of those symbols.

The assumption, therefore, upon which $\sqrt{-1}$ is considered and spoken of as an impossible quantity, is unfounded. All lines drawn from C (Fig. 1.) are as real and possible as CA, which we symbolize by $+1\times a$, or CB, which we symbolize by $-1\times c$. None of them, however, except CA and CB, can be symbolized, as to length and position, by a or c multiplied into either a positive or a negative quantity; since that would be equivalent to saying that they are coincident with CA or CB.

Fig. 1.

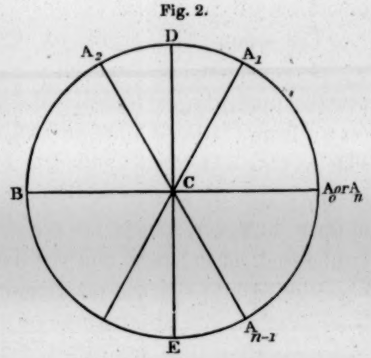


Ordinary algebra, however, has not provided any system of symbols by which these inclined lines may be expressed, both as to length and position, but affords symbols only for the two extreme cases CA and CB. This deficiency Mr Warren has undertaken to supply in his Treatise on the Square Roots of Negative Quantities, published in 1828; and has proposed a system of symbols, which, on the same principle as justifies the use of -1 as the coefficient designating the position of CB, designate as coefficients the position of all lines drawn from C, and making angles with CA.

On some points, however, Mr Warren has been too sparing of his words, and has thus apparently used the common symbols of algebra in a sense very different from their ordinary acceptation. In the following paper I have endeavoured to supply this deficiency of explanation; and then to apply the system of symbols so established to some important problems of goniometry to which, as far as I know, it has not yet been applied. Dr Peacock, in his Treatises on Algebra, has made a somewhat similar use of the coefficients of direction, though arriving at his conclusions by a different route.

II. If from C (Fig. 1.) we draw any number of straight lines in the same plane, such that CA, CA_1, CA_2, &c., shall be continued proportionals, according to Euclid's definition; and make, at the same time, the angles ACA_1, A_1CA_2, A_2CA_3, &c., all equal; then if we call $CA=1$ and $CA_1=a$, CA_2 will equal a^2, $CA_3=a^3$, and so on. The several lines then are arithmetically represented as to their respective lengths by the series 1, or a^0, a^1, a^2, &c. But it is manifest that the several indices which determine the length of the several lines, designate, at the same time, the angles which they make respectively with CA. Thus if a' makes with CA, or unity, an angle ϑ, a^2 makes with CA an angle 2ϑ, a^3 an angle 3ϑ, and so on. And conversely the line which makes with CA an angle $n\vartheta$ is properly represented by a^n. If, instead of calling CA unity, we represent it by R or $R\times 1$, then $CA_1=R\,.\,a^1$, $CA_2=R\,.\,a^2$, and so on.

Fig. 2.



III. If, next, we assume that the several lines CA, CA_1, &c. are all equal, *i.e.*, that they are the consecutive radii of a circle making equal angles with one another (as in Fig. 2.), the first property, proportion, is not thereby destroyed; and we may still properly represent them (beginning with CA_1) by the series a^1, a^2 . . . a^n.

Now let n be a divisor of $2r\pi$; or, ϑ being that angle which each line makes with the succeeding, let $n\vartheta=2r\pi$, or $\vartheta=\frac{2r\pi}{n}$. Then from the last proposition we infer

that n, which is the index of the last term, is also the coefficient of the angle which it makes with that line whose coefficient we assume to be unity, that is, with CA. But $n\vartheta = 2r\pi$, or an integer number of complete circumferences. Hence the radius symbolized by a^n coincides in length and position with the original AC, or $a^n = 1$, therefore $a^1 = 1^{\frac{1}{n}} = 1^{\frac{\vartheta}{2r\pi}}$.

Now we know, on ordinary algebraic principles, that the several nth roots of unity are properly represented by the several terms of the geometric series $a, a^2, a^3 \ldots . a^n$, or 1. Since, then, the two series, first that of the successive radii of a circle making equal angles with one another, and secondly, that of the several nth roots of unity are in symbolism the same, it follows, that, dropping this common symbolism, we may take the several roots of unity to represent the successive radii, and conversely.

If, as before, we take not unity but R for the numerical length of the radius, then $R \,.\, 1^{\frac{\vartheta}{2r\pi}}$ is the expression for that radius which is inclined to that symbolized by $R \times 1$. at an angle ϑ. And as the direction of the radius, or its angularity to the original position is noted by the numerator of the index, we call $1^{\frac{\vartheta}{2r\pi}}$ the *coefficient of direction*. We have thus found a function of the angle of inclination which, being affixed as a coefficient or multiplier to the arithmetical expression for the length of the radius, represents the radius so inclined, both in length and position; and which may be employed according to the ordinary rules of algebraic calculation, to find the length and position of other lines under conditions of relation to it.

These *coefficients of direction*, however, it must be observed, have no quantitative or arithmetical value. Thus $a \,.\, \frac{1+\sqrt{-3}}{2}$, expresses a line whose length is simply a; the coefficient $\frac{1+\sqrt{-3}}{2}$ affecting not the length, but only the direction of the line.

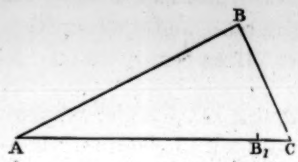
IV. As illustrative of this reciprocal symbolism, let us suppose that the successive radii are two in number, or, in other words, that a radius revolving round C takes only one fixed position, and makes only two equal angles before it returns to its original position (Fig. 2). Then the circumference is divided into two equal parts, AB is the diameter, and if $CA = 1$, $CB = -1$. In this case $n = 2$, therefore $a^2 = 1$ or $a^2 - 1 = 0 \therefore a = \pm 1$. But the radii being a, a^2, a must evidently be -1, and $a^2 = +1$.

Next let the circumference (Fig. 2.) be divided into four equal parts, then CA, CD, CB, CE are the four roots of the equation $a^4 - 1 = 0$. But these roots are ± 1 and $\pm\sqrt{-1}$.

Here CA and CB are, by Art. 1, symbolized by $+1$ and -1 respectively; therefore CD and CE must be symbolized by $+\sqrt{-1}$ and $-\sqrt{-1}$. It is, however, quite optional which direction from C we consider positive, whether in the horizontal or perpendicular line.

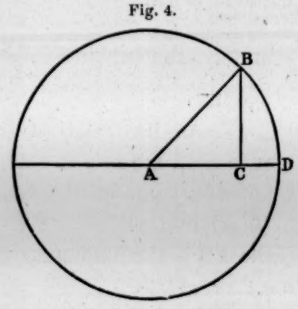
V. It appears from the foregoing propositions, that if a line is presented to us under the symbol $a \cdot 1^{\frac{\vartheta}{2r\pi}}$, we know both its length and the angle ϑ which it makes with a given line whose coefficient of direction we assume to be unity, and which, therefore, we symbolize by a simply. The symbol $a \cdot 1^{\frac{\vartheta}{2r\pi}}$, therefore, represents the actual transference of position in space which a point would undergo by moving from the one extremity of the line to the other, as from A to C (Fig. 3.). But it is clear, also, that if a point be supposed to be removed from A to B, and then from B to C, the actual transference in space, though not the distance travelled, would be the same as if the transference had been direct from A to C. Therefore the symbol which properly represents the one transference, must be symbolically equal to the sum of the two symbols which respectively represent the other two transferences, or AC $\times$ its coefficient of direction $=$ AB $\times$ its coefficient of direction $+$ BC into its coefficient of direction.*

Fig. 3.



This fundamental proposition is given by Mr WARREN as a definition, That the sum of any two lines making an angle with one another is the diagonal of their parallelogram completed. Even in this startling form, it is only the general assertion of a proposition, particular cases of which are admitted, when we say (Fig. 3.) that $AB_1 + B_1C = AC$, or that $AC + CB_1 = AB_1$. By such assertions we really mean that if a point moves from A to B_1, and then from B_1 to C, the whole transference in space will be represented by the sum $AB_1 + B_1C$; and that if the point moves from A to C, and then from C to B_1, the whole transference is expressed by the sum $AC + CB_1$, which is the same thing as the arithmetical difference $AC - B_1C$.

Fig. 4.



As examples to elucidate this proposition, let us take (Fig. 4.) an isosceles right-angled triangle

* This appears to be the view taken by Sir W. HAMILTON, in the first of his series of papers on Symbolical Geometry, printed in the Cambridge and Dublin Mathematical Journal. He there says, "This symbolic sum of lines represents the total (or final) effect of all those successive rectilineal motions, or translations in space, which are represented by the several summands."

ACB. If we call AB the radius or hypothenuse, a, then each of the sides AC CB is in length $\frac{a}{\sqrt{2}}$, and AB (being inclined at an angle of 45° to AD, which we assume as the original position of the radius) is symbolized by $a \times 1^{\frac{45}{360}} = a \times 1^{\frac{1}{8}} = a \cdot \frac{1+\sqrt{-1}}{\sqrt{2}}$. But $\mathrm{AC}=\frac{a}{\sqrt{2}}$, CB being perpendicular to the original position equals $\frac{a}{\sqrt{2}} \cdot \sqrt{-1}$. (Prop. IV.) Therefore $\mathrm{AC}+\mathrm{CB}=a \cdot \left[\frac{1}{\sqrt{2}}+\frac{\sqrt{-1}}{\sqrt{2}}\right]=a \cdot \frac{1+\sqrt{-1}}{\sqrt{2}}=\mathrm{AB}$.

2. Let BAC represent a right-angled triangle whose angle at $\mathrm{A}=60^{\circ}$, then AB in length and direction $=a \cdot 1^{\frac{60}{360}}=a \cdot 1^{\frac{1}{6}}=a \cdot \frac{1+\sqrt{-3}}{2}$, $\mathrm{AC}=\frac{a}{2}$, CB = in length $\frac{\sqrt{3}}{2}$, and therefore in length and direction jointly $a \cdot \frac{\sqrt{3} \cdot \sqrt{-1}}{2} = a \cdot \frac{\sqrt{-3}}{2}$

$$\therefore \quad \mathrm{AC}+\mathrm{CB}=\frac{a}{2}+a \cdot \frac{\sqrt{-3}}{2}=a \cdot \frac{1+\sqrt{-3}}{2}=\mathrm{AB}.$$

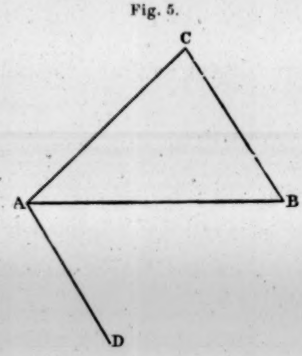
3. Let the triangle (Fig. 5.) be equilateral, and AB be taken as the original position. Let $\mathrm{AB}=a$, $\mathrm{AC}=a \cdot 1^{\frac{1}{6}}$, $\mathrm{CB}=a \cdot 1^{-\frac{1}{6}}$

$$\therefore \quad \mathrm{AC}+\mathrm{CB}=a\left[1^{\frac{1}{6}}+1^{-\frac{1}{6}}\right]=a \cdot \left[1^{\frac{1}{6}}+\frac{1}{1^{\frac{1}{6}}}\right]=a\left[\frac{1^{\frac{1}{3}}+1}{1^{\frac{1}{6}}}\right]$$

$$=.a \left[\frac{-1+\sqrt{-3}}{2}+1\right] \times \frac{2}{1+\sqrt{-3}}=a\left[\frac{1+\sqrt{-3}}{2} \times \frac{2}{1+\sqrt{-3}}\right]=a=\mathrm{AB}.$$

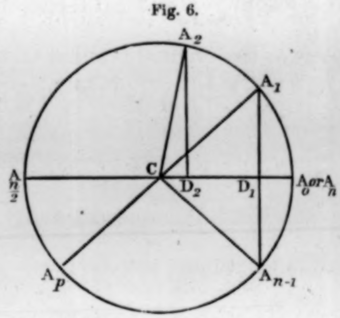
VI. In the foregoing Propositions and Examples, it has been assumed that we know not only the several nth roots of unity, but also their proper order, that is, the order in which, as coefficients, they express the radii drawn so as to make angles ϑ, 2ϑ, 3ϑ, &c., with the original radius. But when by any analytical process we find the roots of $x^n-1=0$, we procure the symbolical representatives of these radii in no determinate order. To discover this order, we must observe that two roots are always of the form $a \pm \sqrt{-b}$; comparing which expression with figure 6, it is evident that a is the part symbolical of the cosine, and $\sqrt{-b}$ the part symbolical of the sine, because it is affected by the coefficient $\sqrt{-1}$, and is therefore perpendicular to the original radius. It is clear, then, that in the general expression $a \pm \sqrt{-b}$, the sign + belongs to those radii which lie in the upper half of the circle, and − to those which lie in the lower half; and that the two radii whose symbols differ only in the

Fig. 5.



sign of $\sqrt{-b}$, are at equal angles to the original radius, in different directions, that is, on different sides of it.

Fig. 6.



Again, of those roots which symbolize the radii in the upper half of the circle, that which has a, representing the cosine greatest, is the nearest to the original radii. Thus the roots of $n^6-1=0$, in the order given by Dr PEACOCK, Alg. ii., p, 128, are 1, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$ -1, $\frac{1-\sqrt{-3}}{2}$, $\frac{1+\sqrt{-3}}{2}$. To arrange these in their proper order, if $+1$ be placed first, then -1, as having no sinal part, and being therefore, neither in the upper nor lower half, must stand in the middle of the remaining roots. Next these are two roots, $\frac{1+\sqrt{-3}}{2}$ and $\frac{-1+\sqrt{-3}}{2}$, each having the sinal part $+$, which must be arranged in this order, because the sign of 1 in the former indicates that the cosine is in CA, and in the latter in $CA_{\frac{n}{2}}$. Finally, considering those roots of which the sinal part is minus; we must place them in the order $\frac{-1-\sqrt{-3}}{2}$, $\frac{1-\sqrt{-3}}{2}$, because they are thus equidistant from unity with $\frac{-1+\sqrt{-3}}{2}$ and $\frac{1+\sqrt{-3}}{2}$. Hence the roots in their proper sequence are

$$1,\ \frac{1+\sqrt{-3}}{2},\ \frac{-1+\sqrt{-3}}{2},\ -1,\ \frac{-1-\sqrt{-3}}{2},\ \frac{1-\sqrt{-3}}{2},$$

symbolizing severally the radii drawn to the extremities of the arcs

0 or 360°, 60°, 120°, 180°, 240°, 300°.

VI. It appears from Props. IV., V.. that the radius drawn to the extremity of an arc ϑ, is properly expressed by $1^{\frac{\vartheta}{2\pi}}$, and this again by $a\pm\sqrt{-b}$, where a is what is called in trigonometry the cosine of ϑ, and $\sqrt{b}$ the sine.

Now let CA_1 (Fig. 6.) make with CA an angle ϑ, CA_2 an angle 2ϑ . . . CA_p an angle $p\vartheta$.

Then
$$CA_1=CD+\sqrt{-1}\,.\,DA_1=\cos\vartheta+\sqrt{-1}\,.\,\sin\vartheta$$
$$CA_p=\cos p\vartheta+\sqrt{-1}\,.\,\sin p\vartheta.$$

But by Prop. II. $CA_p=\overline{CA_1}|^p=(\cos\vartheta+\sqrt{-1}\,.\,\sin\vartheta)^p$

$$\therefore\quad (\cos\vartheta+\sqrt{-1}\,.\,\sin\vartheta)^p=\cos p\vartheta+\sqrt{+1}\,.\,\sin p\vartheta,$$

which is DEMOIVRE's Theorem.

COR. If $p\vartheta=2\pi$, $\cos p\vartheta+\sqrt{-1}\,.\,\sin p\vartheta=1$.

Hence $(\cos\vartheta+\sqrt{-1}\,.\,\sin\vartheta)$, $(\cos 2\vartheta+\sqrt{-1}\,.\,\sin 2\vartheta)$ &c. represent the several pth roots of unity. If, instead of the order ϑ, 2ϑ, 3ϑ, &c., we arrange the several angles thus in pairs ϑ and $\overline{p-1}\,.\,\vartheta$, 2ϑ and $\overline{p-2}\,.\,\vartheta$, then the several expressions for x minus the several pth roots of unity, or the several simple factors of the equation $x^p-1=0$, taken in pairs corresponding to the above, will be

$$(x-\cos\vartheta-\sqrt{-1}\,.\,\sin\vartheta) \text{ and } (x-\cos\overline{p-1}\,.\,\vartheta-\sqrt{-1}\,.\,\sin\overline{p-1}\,.\,\vartheta),$$

the latter of which equals $(x-\cos\,.\,\overline{p\vartheta-\vartheta}-\sqrt{-1}\,.\,\sin\overline{p\vartheta-\vartheta})$

$$=x-\cos\overline{2\pi-\vartheta}-\sqrt{-1}\,.\,\sin\overline{2\pi-\vartheta}=x-\cos\vartheta+\sqrt{-1}\,.\,\sin\vartheta.$$

In the same way the next pair must be

$$(x-\cos 2\vartheta+\sqrt{-1}\sin 2\vartheta) \text{ and } (x-\cos 2\vartheta-\sqrt{-1}\,.\,\sin 2\vartheta), \text{ and so on.}$$

If these several pairs be next multiplied together so as to produce the quadratic factors of $x^n-1=0$, we obtain the products $(x^2-2x\cos\vartheta+1)$, $(x^2-2x\,.\,\cos 2\vartheta+1)$ &c. And if it be remembered that in every case $x-1=0$ is a factor; and that if p be even, $x-1$ and $x+1$ are simple factors, and consequently x^2-1 a quadratic factor; therefore if p be even,

$$x^p-1=(x^2-1)\,.\,(x^2-2x\cos\vartheta+1)\,.\,(x^2-2x\,.\,\cos 2\vartheta+1)\text{ \&c. to } \frac{p}{2} \text{ terms.}$$

But if p be odd,

$$x^p-1=(x-1)\,.\,(x^2-2x\cos\vartheta+1)\,.\text{ \&c. to } \frac{p+1}{2} \text{ terms.}$$

Where ϑ, it may be observed, equals $\frac{2\pi}{p}$.

VIII. From these fundamental propositions, Mr WARREN, in his Treatise on Negative Roots, has deduced—

1. The value of each side of a triangle in terms of the other sides and angles. (§ 141.)

2. That the three angles of a triangle are equal to two right angles. (§ 142.)

3. That the sides are respectively proportional to the sines of the opposite angles. (§ 143.)

4. That $\cos A=\frac{b^2+c^2-a^2}{2bc}$. (§ 144.)

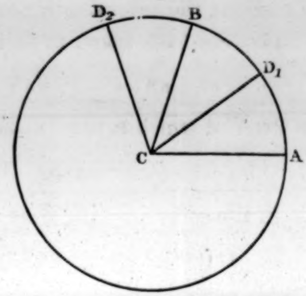
He then asserts, that from these and the preceding propositions, all the formulæ of plane trigonometry may easily be deduced. In the following propositions, I have applied his principles to the solution of some of the most simple, and to some of comparatively the more difficult problems usually given in elementary books of trigonometry.

IX. $$\sin(A+B) = A \times \cos B + \cos A \times \sin B$$

$$\cos(A+B) = \cos A \times \cos B - \sin A \, . \, \sin B.$$

Let arc AB (Fig. 7.) $=A$, BD_2 and AD_1 each $=B$.

Fig. 7.



Then by Prop. III., $CB = r \, . \, 1^{\frac{A}{2\pi}}$,

$$CD_1 = r \, . \, 1^{\frac{B}{2\pi}}, \quad CD_2 = r \, . \, 1^{\frac{A+B}{2\pi}}$$

$$\therefore \quad CD_2 = r \times 1^{\frac{A}{2\pi}} \times 1^{\frac{B}{2\pi}}$$

But Prop. VII.,

$$1^{\frac{A}{2\pi}} = \cos A + \sqrt{-1} \, . \, \sin A$$

$$1^{\frac{B}{2\pi}} = \cos B + \sqrt{-1} \, . \, \sin B$$

$$\therefore \quad 1^{\frac{A+B}{2\pi}} = \cos A \times \cos B - \sin A \, . \, \sin B + \sqrt{-1} \, . \, (\sin A \, . \, \cos B + \cos A \, . \, \sin B$$

but $$1^{\frac{A+B}{2\pi}} = \cos \overline{A+B} + \sqrt{-1} \, . \, \sin \overline{A+B}$$

Equating, then, the possible and impossible, or, more properly, the sinal and cosinal, parts of these equal forms

$$\cos A \times \cos B - \sin A \, . \, \sin B = \cos \overline{A+B}$$

and $$\sin A \times \cos B + \cos A \, . \, \sin B = \sin \overline{A+B}.$$

This demonstration is the same in principle, and nearly the same in detail, as that given by Dr PEACOCK, in his Algebra, vol. i., p. 392. In his 2d volume, Dr PEACOCK goes more fully into the consideration of the roots of unity as coefficients of direction. Yet there he proves these propositions, not upon that consideration, but by the ordinary geometrical method.

DEF. It should be observed that in the following propositions, a line expressed by letters simply as AB, must be understood as considered in respect both of length and direction; while by the same letters in brackets, thus (AB), is understood the same line in regard to its length only. Thus, if ϑ be the angle which AB makes with unity, $(AB) \, . \, 1^{\frac{\vartheta}{2\pi}} = AB$.

X. In any right-angled triangle, the sum of the squares of the sides is equal to the square of the hypothenuse.

Let CA (Fig. 6) $=r$, then $CA_1 = r \, . \, 1^{\frac{\vartheta}{2\pi}}$, and $CA_{n-1} = r \, . \, 1^{-\frac{\vartheta}{2\pi}}$

$$\therefore \quad CA_1 \times CA_{n-1} = r^2 \times 1^{\frac{\vartheta}{2\pi}} \times \frac{1}{1^{\frac{\vartheta}{2\pi}}} = r^2$$

Also $\quad CA_1=(CD_1)+\sqrt{-1}\,.\,(D_1A_1)$

$$CA_{n-1}=(CD_1)-\sqrt{-1}\,.\,(D_1A_1) \text{ for } (D_1A_1)=(D_1A_{n-1})$$

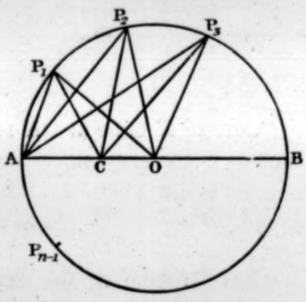
$$\therefore \quad CA_1 \times CA_{n-1}=(CD_1{}^2)+(D_1A_1)^2 \text{ which } \therefore = r^2=(CA)^2$$

or its equivalent in area $(CA_1)^2$.

XI. Cotes' *Properties of the Circle.*

Let the circumference of the circle be divided into n equal parts; and to the extremities of these let lines be drawn from the centre (Fig. 8), as OP_1, OP_2, &c., and from any other point C in the diameter. Then

Fig. 8.



$$CP_1=OP_1-OC,\ CP_2=OP_2-OC,\ \&c.$$

$$\therefore \quad CP_1 \times CP_2 \times CP_3 \ \ldots\ldots\ CP_n$$

$$=\Sigma_n\,.\,(OA)^n-\Sigma_{n-1}(OA)^{n-1}\ \ldots\ldots\ \pm OC^n$$

Where Σ_n is the product of all the coefficients of direction for OP_1, OP_2, &c., Σ_{n-1}, the sum of these coefficients taken $\overline{n-1}$ together, and so on. But these coefficients (Prop. III.) are also the values of $1^{\frac{1}{n}}$, or the roots of the equation $x^n-1=0$. Now the product of the roots of this equation with their signs changed is -1, and Σ_n is the product with the signs unchanged.

Therefore if n be even, $\Sigma_n=-1$, and, if n be odd, $\Sigma_n=+1$; and in either case, Σ_{n-1}, Σ_{n-2}, &c., each $=0$.

Hence $CP_1 \times CP_2 \ \ldots\ldots \times CP_n=\pm(OA)^n \pm (OC)^n$; the upper signs being used when n is even, the lower when n is odd.

But CP_1 CP_2, &c., represent the lines considered in relation both to length and direction; therefore, to change the equation into one in which the length only of these lines shall be expressed, we must divide the first side, or multiply the second by the product of all their coefficients of direction.

If n be even, the several pairs, as CP_1, CP_{n-1}, are evidently of the form

$$(CP_1)\,.\,1^{\frac{\theta}{2\pi}} \text{ and } (CP_{n-1})\,.\,1^{-\frac{\theta}{2\pi}} \ \therefore\ CP_1 \times CP_{n-1}=(CP_1)\times(CP_{n-1})$$

and the same is true for every pair except $CA=(CA)\,.\,+1$ and $CB=(CB)\,.\,-1$

$$\therefore \quad (CP_1)\times(CP_2)\ \ldots\ldots\ CP_n=[-OA^n+OC^n]\times -1=OA^n-OC^n.$$

If, again, n be odd, the several pairs remain as before, only, no P falling upon $B,-1$ is not a coefficient of direction:

$$\therefore \quad (CP_1 \times (CP_2)\times \&c., =OA^n-OC^n \text{ as before.}$$

COR. 1. If C be on the opposite side of O from A, the other conditions remaining the same, OC is negative. If n be even, the expression deduced in the proposition remains unchanged. But if n be odd, $(CP_1)\times(CP_2)\times$ &c., $=OA^n+OC^n$. And here it may be remarked, that when lines, as OA are in the original direction, since the coefficient of direction in that case is unity, it is immaterial whether we write OA or (OA).

Ex. Let $n=3$ and $OC=\frac{1}{2}$

then $$(AC)=\tfrac{3}{2},\ (CP_1)=(CP_2)=\frac{\sqrt{3}}{2}$$

$$\therefore \quad (CA)\,.\,(CP_1)\times(CP_2)=\tfrac{3}{2}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2}=\tfrac{9}{8}=1+\tfrac{1}{8}$$

$$=1^3+\overline{\tfrac{1}{2}}|^3=OA^3+OC^3.$$

COR. 2. If C be in OA produced, the reasoning and the result will be the same as in the proposition; only, that now CA and CB being of the same affection, -1 is not a divisor of the second number of the equation, and

$$(CP_1)\times(CP_2)\times\&c.,=(OC)^n-(OA)^n.$$

XII. If from A, the extremity of the diameter (Fig. 8), the circumference be divided into n equal parts, and lines be drawn to their several extremities from A, then

$$(AP_1)\times(AP_2)\ \ldots\ .\ (AP_{n-1})=n\,.\,CA^{n-1}$$

As in the preceding proposition $AP_1=CP_1-CA$, $AP_2=CP_2-CA$, and so on. Therefore $AP_1\times AP_2\times\ \ldots\ .\ AP_{n-1}=\overline{CP_1-CA}\times\overline{CP_2-CA}\times$ &c., to $n-1$ factors

$$=R^{n-1}\,.\,\left\{S_{n-1}-S_{n-2}\ \ldots\ .\ \pm S_1\pm1\right\}$$

where S_1, S_2, are the sum, sum of products 2 and 2, &c., of all the values of $1^{\frac{1}{n}}$ except unity, there being no line drawn from A to the circumference in the direction CA. S_1, S_2, &c., are, therefore, the coefficients of the equation $\frac{x^n-1}{x-1}$, or of $x^{n-1}+x^{n-2}\ \ldots\ .\ +1=0$, with the signs changed for the products of odd numbers of roots, unchanged for even ones.

If, therefore, $n-1$ be even, $S_{n-1}=+1$, $S_{n-2}=-1$, and so on.

If $n-1$ be odd, $S_{n-1}=-1$, $S_{n-2}=+1$, and so on.

$\therefore\ \ AP_1\times AP_2\times$ &c., $\ldots\ .\ =\,.\ R^{n-1}\times\pm\{1+1+1$ to n terms$\}=\pm n\,R^{n-1}$ according as $\overline{n-1}$ is even or odd.

If $\overline{n-1}$ be even, then $AP_1+AP_2\times$ &c. $=(AP_1)\times(AP_2)$ &c., the several pairs of coefficients of direction giving unity as their product.

If $n-1$ be odd, then the several pairs give as before the product unity; but there remains the factor $-\mathrm{AB}$, which has for its coefficient -1.

Therefore, in either case, $(\mathrm{AP}_1) \times (\mathrm{AP}_2) \ldots\ldots (\mathrm{AP}_{n-1}) = n\,\mathrm{R}^{n-1}$.

XIII. The symbolism employed in the foregoing propositions appears to be applicable to Plane Trigonometry in all its parts. To the elementary propositions of Geometry it is either inapplicable, or applicable by processes and considerations unsuitable to the demonstration of elementary truths. Thus, if by this method we undertake to prove that the angles at the base of an isosceles triangle are equal to one another, we have $(\mathrm{AC}) = (\mathrm{BC})$. (Fig. 5.)

But $\qquad \mathrm{AC} = (\mathrm{AC}) \cdot 1^{\frac{\mathrm{A}}{2\pi}} = (\mathrm{AC}) \cdot [a + \sqrt{-b}]$

$\qquad \mathrm{CB} = \mathrm{AD} = (\mathrm{AC}) \cdot 1^{-\frac{\mathrm{B}}{2\pi}} = (\mathrm{AC}) \cdot [a' + \sqrt{-b'}]$

But $\qquad \mathrm{AC} + \mathrm{CB} = \mathrm{AB}.$

$\therefore$ $(\mathrm{AC}) \cdot [a + a' + \sqrt{-b} + \sqrt{-b'}] = \mathrm{AB}$ a positive quantity; consequently the impossible or sinal parts of the coefficient of direction must destroy one another, or $\sqrt{-b} = -\sqrt{-b'}$ or $b = -b'$. Therefore the angles A and B have their sines equal in length, but of different affections. The angles themselves, therefore, being together less than π, are geometrically equal to one another.

Cor. Much in the same way we might prove that in every triangle the greater angle has the greater side opposite to it; and, conversely, that the greater side has the greater angle opposite to it.

XXIII.—*On the Reaction of Natural Waters with Soluble Lead Salts.* By Arthur Connell, Esq., F.R.S.E., *Professor of Chemistry in the University of St Andrews.*

(Read 19th January 1846.)

In a former communication to the Society, I noticed a reaction presented by all spring, well, and river waters which I had examined, that, even after being boiled, they yielded, with acetate of lead, a precipitate readily soluble, in whole or in great part, in acetic acid. This easy solubility in acetic acid shewed that the precipitate was neither a sulphate nor a phosphate, and the comparatively slight action of nitrate of silver proved that it was not a chloride. There seemed, therefore, to remain only the conclusion that it was either a carbonate, or was due to organic matter. The former alternative, of course, depended on whether the solution in acetic acid was attended with effervescence or not; and as this seemed usually not to be the case, and as, on decomposing some of the precipitate by sulphuretted hydrogen, some organic matter in solution was obtained, the conclusion seemed to be, that the appearance was caused by organic matter, probably of the nature of the crenic and apocrenic acids of Berzelius. I have since, however, found that by very careful observation, effervescence may be noticed during the solution of the precipitate more frequently than I at first supposed. It is not so easy as might be imagined to determine this point. If acetic acid is added before the precipitate has subsided, no effervescence can be noticed, in almost any case, from the water dissolving the carbonic acid evolved and diffused through the whole liquid. And even when allowed to subside, and the greater part of the liquid is decanted, the addition of acetic acid not unfrequently causes solution without apparent effervescence. The cautious addition, however, of a heavier acid, such as the nitric or even the muriatic, after allowing the lead precipitate to collect at the bottom, and decanting the greater part of the liquid, seldom fails to shew the effervescence where a carbonate is really present.

In so far, then, as the precipitate is dissolved by acids with effervescence, we may conclude that it has been caused by some carbonate remaining dissolved after boiling the water; and in so far as the solution may not exhibit effervescence, we may conclude that it is due to organic matter, provided silver salts do not indicate the presence of a sufficient quantity of chlorides, and provided acetic acid instantly causes solution in whole or great part. In some instances I have found that acetate of lead does not yield a precipitate, *unless* the water has been previously boiled, a circumstance obviously due to excess of carbonic acid retaining the carbonate of lead in solution. In regard to chlorides, I have never met

with any spring, well, or river water, not coming under the denomination of a mineral water, which contained so much of any chloride as to be indicated by a lead salt. The chloride of lead is too soluble to become visible, unless where the contamination is considerable. The portion of the precipitate not soluble in acetic acid is usually due to the presence of some sulphate.

Taking the fact as I have now ascertained it to be, that natural waters which have passed through the strata or soils of the earth, *i. e.*, well, spring, and river waters, very commonly or invariably yield, even after boiling, and filtering, if necessary, a greater or less amount of precipitate with acetate of lead, readily dissolved, in whole or in part, by acetic acid with effervescence; in other words, that such natural waters contain frequently or invariably, even after boiling, one or more dissolved carbonates, the question arises, What is the nature of such carbonate? As these waters, when they have been much concentrated after being boiled and filtered, and have then been made up to their former bulk by distilled water, are found to have lost their power of shewing the same phenomena as before with lead salts, and to have deposited carbonate of lime during concentration, the effect must have been due to this carbonate of lime whilst held in solution. The farther question, therefore, arises, how this carbonate of lime came to be dissolved?

I tried to ascertain whether water would dissolve carbonate of lime in its nascent state as precipitated by boiling a solution in excess of carbonic acid. A current of carbonic acid was passed through lime water prepared with distilled water, until the precipitate at first formed was redissolved. The solution was then boiled for a similar short time as in the original experiments, and filtered. It was then found to be affected only very feebly either by oxalate of ammonia or by acetate of lead; the action being not at all equal to that produced on boiled natural waters by these reagents. The carbonate of lime in the act of precipitation by boiling, had evidently not been dissolved except in very insignificant quantity by the water. I then, before boiling the solution, exposed it for a day to the air in an evaporating basin, after keeping it for some days in a close vessel; but did not find that the quantity of carbonate retained after ebullition, short subsidence, and filtration, was increased.

As it was possible that some of those saline matters contained in natural waters might promote the solubility of carbonate of lime, minute quantities of solutions of muriate of lime, sulphate of lime, muriate of magnesia, and chloride of potassium were added to lime water prepared with distilled water. A current of carbonic acid was then conducted through the liquid so as to redissolve the precipitate which it at first caused. The liquid, after ebullition, short subsidence, and filtration, was found scarcely to be affected by acetate of lead; and any feeble deposit formed was not soluble in acetic acid, being sulphate of lead due to the sulphatesw hich had been added.

It thus seemed evident that by the aid of carbonic acid no sufficient quantity

of carbonate of lime can be dissolved, independently of the continued presence of the free acid, to cause the appearances referred to.

I next tried the solvent action of water alone on finely divided carbonate of lime. Distilled water, which had been boiled and cooled, was left in contact with marble in impalpable powder for several days in a close vessel. It was then found by the action both of acetate of lead and of oxalate of ammonia, that rather more carbonate of lime had been taken up by the pure water than was left in solution after boiling the carbonated water, but still that the amount was considerably less than the reactions which have been referred to, indicate in ordinary natural waters; and it is remarkable that the effect of the lead salt is usually more decided than that of the oxalate.

I incline, therefore, to think, that the carbonate of lime present, in such circumstances as have been described, has a different origin, viz., Double Decomposition, between a lime-salt and a carbonated alkali; as it would seem that the carbonate of lime formed is, in this kind of nascent state, dissolved more readily than when precipitated by boiling from a carbonic solution. The following experiments illustrate this origin of the reaction. To half an ounce of distilled water, a few drops of a solution of sulphate of lime in water were added. A single drop of solution of chloride of calcium, and a single drop of solution of carbonate of potash were then added. The liquid remained quite transparent, and did not affect turmeric or cabbage paper. When boiled it still remained transparent. When a drop of solution of acetate of lead was added to a portion of this liquid, either before or after boiling, a considerable white cloud was formed, which disappeared on the addition of a drop of acetic acid. Thus was the reaction of the spring waters exactly imitated. I at first inclined to think that in such cases no actual double decomposition ensued until the liquid was concentrated by heat, and that the action on the lead-salt was due to the carbonated alkali present. But farther experiments lead me to believe that the carbonate of lime is actually formed, at least to a considerable extent, and then dissolved by the water; for, if a couple of drops of solution of chloride of calcium and a drop of solution of carbonate of potash be added to a few drops of distilled water, muddiness will be produced; and this will disappear when half an ounce of distilled water is shaken with the mixture, without any deposit being formed by rest. The solution, however, of the carbonate of lime is dependent on the action of the water taking place either on the nascent salt, or at least immediately after its formation; for I found that when carbonate of lime, precipitated by double decomposition, was collected on a filter, washed, and allowed to stand some minutes, and then left in contact all night with boiled and cooled distilled water, acetate of lead had only a very feeble effect on the liquid.

It was, of course, necessary, in order to establish this view, to ascertain that those natural waters which exhibit the reaction referred to, actually contain

alkaline matters. This was, accordingly, done in numerous instances by ordinary methods. Although the potash or soda present may have been originally dissolved as a carbonate; yet we, of course, ultimately obtain it on evaporation, as a chloride or sulphate, through double decomposition with the lime or magnesian salts present; or through the stronger affinity of the acids of these salts, if their earths have been previously removed by chemical means. In no instance of a natural water which gave the reaction with lead salts, did I fail to detect either potash or soda, or both; and it ought to be recollected that a very minute quantity of either is sufficient. It will be found that one drop each of solutions of carbonate of potash, of sulphate of magnesia, and of chloride of calcium, added to several ounces of distilled water, will produce the reactions referred to with lead salts and acetic acid.

If these views are well-founded, it is evident that lead salts become a probable indication, at least where their effect is considerable, of the presence of alkalies in natural waters. And, in general, we may conclude, that if after boiling, and filtration if necessary, any water yields a considerable cloud with acetate of lead, readily soluble by adding a drop or two of acetic acid, the cause will be either carbonate of lime, probably due to double decomposition, or it will be organic matter, if any such matter precipitable by lead salts is present in sufficient quantity.* In so far as it is dissolved by an acid, after subsidence, with effervescence, it will be due to the former cause; in so far as, without effervescence, to the latter.

It seems, at all events, evident from the experiments which have been detailed, that the carbonate of lime present has not owed its presence to the solvent agency of carbonic acid, even when first taken up.

It is plain, that the carbonate of lime thus held dissolved by spring waters, from whatever source it may be obtained, must be of considerable importance in the economy of nature in furnishing a supply, through the intervention of these waters, of that lime which is so essential a constituent, in its various states of combination, of the inorganic portion of plants. This will hold whether such waters are applied to the land in the way of irrigation, or by the more slow processes of natural infiltration.

* Dr Christison informs me that moss-water is not precipitated by acetate of lead. This, I have no doubt is a correct observation; but still other states of organic matter may occasion a precipitate. The crenic and apocrenic acids are both known to precipitate lead salts.

Although fluorine is now known to be occasionally present in ordinary natural waters, and although fluoride of lead is sparingly soluble in water, yet I am not aware that fluorine is ever present in such quantity in such waters as to affect lead salts; and, if it were, acetic acid might very likely not dissolve the precipitate. Dr Wilson mentions that fluoride of barium is less soluble in acids than carbonate or phosphate of barytes.

POSTSCRIPT.

Since this paper went to press, I have ascertained that the town water of St Andrews, which is one of those which gives the reaction referred to with lead salts, yields by evaporation, after having been boiled and filtered, $\frac{1}{31323}$ of its weight of carbonate of lime. Other waters may of course contain more. I also observe, that FRESENIUS found, that when distilled water was boiled a long time (probably, from the context, several hours) with freshly precipated carbonate of lime, so as to form a saturated hot solution, and this solution was then kept for four weeks at common temperatures, in contact with undissolved carbonate of lime, under frequent agitation, it yielded by evaporation $\frac{1}{10601}$ of its weight of carbonate of lime.—*Liebig's Annalen, July* 1846. In so far as regards spring waters, it is unnecessary to say, that Nature does not take such pains to charge them with lime. The method suggested above seems a more simple one, and may often be as effectual, possibly even more so; when the still more simple means of free carbonic acid are not brought into play. From the experiments of FRESENIUS, it appears, that carbonate of lead is much less soluble in water than carbonate of lime, viz., in 50,551 parts, which is quite conformable to the results above stated.

XXIV.—*On certain Products of Decomposition of the Fixed Oils in contact with Sulphur.* By THOMAS ANDERSON, Esq. M.D., F.R.S.E., *Lecturer on Chemistry, Edinburgh.*

(Read 19th April 1847.)

Numerous researches have established as a general rule that the products of the decomposition of organic substances vary with the circumstances of the experiment, and the nature of the agents under the influence of which it is performed. If, for instance, we examine the action of heat alone, we find it causing a set of decompositions specially characterised by the evolution of carbonic acid, formed by the union of part of the carbon of the substance with the whole or part of its oxygen; and this action is rendered more definite, and the number of the products circumscribed by all circumstances facilitating the formation of carbonic acid, such as the presence of a base, which will even cause its evolution when heat alone is incapable of producing decomposition. Acids, on the other hand, have a precisely opposite effect, they, in some instances, altogether prevent the formation of carbonic acid, and cause the oxygen to exert its action on the hydrogen of the compound, and to eliminate one or more atoms of water which do not generally exist ready formed in it.

In these particular instances decomposition takes place at the expense of the constituent atoms of the compounds themselves, the extraneous substances serving merely as disponents to the oxidation, in the one case of part of their carbon, in the other of their hydrogen; but there is another class of agents which, besides eliminating one or more substances, are capable at the same time of entering into union with the residual atoms, and forming a new derivative of the original compound. The best investigated of this class of agents are chlorine, bromine, nitric acid, and ammonia, the three former of which exert their action on the hydrogen, the latter on the oxygen of the substance, and form compounds the complete investigation of which is important, not merely in a purely chemical point of view, but also from the light which they seem likely to throw on the general question of the atomistic constitution of matter. In fact, the great object of the researches of organic chemistry at the present moment is that of developing the relations which the individual atoms bear to the molecules of their compound, by a knowledge of which we hope eventually to arrive at some definite conclusions with regard to the mode in which the elementary atoms are grouped together in a complex molecule. Almost all the scanty information which we possess on this subject has been derived from investigating the products of the action of different agents upon organic substances; and it is sufficiently obvious, that the more varied the circumstances, and numerous the points of view under which these re-

actions can be examined, so much the more likely are we to arrive at definite results.

It was the consideration of these points which led me to undertake an investigation into the nature of the action of Sulphur in the free state upon organic compounds, a subject hitherto totally uninvestigated, unless we except the curious researches of ZEISE* on the simultaneous action of ammonia and sulphur upon acetone, which yields a variety of remarkable products, the properties of which he has described, without however determining their constitution. The results at which I have already arrived in these researches are contained in the following pages. They are, however, to be considered only as the commencement of the investigation; and I am desirous of submitting them to the Society even in their present very imperfect state, as it is impossible to fix a period within which a series of researches, surrounded by so many difficulties, can be completed. No one who has not been specially occupied with such experiments can have any conception of the numerous sources of annoyance which they present, and of the expenditure of time and labour which is necessary for their performance. Indeed, I have more than once felt inclined altogether to abandon a subject occupying so much time in proportion to the results obtained, and the completion of which is further protracted by the nauseous odour of the compounds, which is so disgusting that it is impossible to pursue the investigation for any length of time continuously.

At the commencement of these researches I endeavoured to examine the action of sulphur upon some of the simpler organic compounds, in the hope of arriving at results of corresponding simplicity. My expectations, however, were disappointed, and I was obliged to have recourse to the fixed oils, on which sulphur has been long known to exert an action; the product obtained by heating together olive oil and sulphur until an uniform balsam-like substance was formed, having been employed in medicine by the older physicians under the name of the Balsam of Sulphur.

The phenomena which manifest themselves during the mutual action of Sulphur and a Fixed Oil are these:—At the first application of heat the sulphur melts and forms a stratum at the bottom of the oil; but as the temperature rises it slowly dissolves, with the formation of a thick viscid fluid of a dark red colour. As the heat approaches that at which the oil undergoes decomposition when heated alone, a violent action takes place attended by the evolution of sulphuretted hydrogen in such abundance that the viscid mass swells up and occupies a space many times its original bulk. If at this point the mixture be allowed to cool, it concretes into a tough sticky tenaceous mass, adhering strongly to the fingers, and having a disagreeable sulphureous odour; if, however, the heat be

* Förhandlingar vid de Skandinaviska Naturforskarnes tredje möte, p. 303.

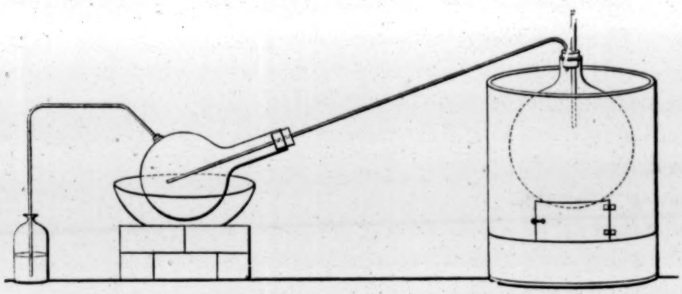
sustained, the frothing and evolution of sulphuretted hydrogen continue, and at the same time, an oil of a peculiar disgusting odour, resembling that of garlic, but more disagreeable, passes into the receiver.

In the investigation of the products of this action, the first and most essential step was to determine the particular constituents of the oil from which they are derived. In order to do this, it was necessary to examine separately the action of sulphur upon each of its components. I commenced, therefore, by making use of stearic acid, which can be readily obtained in a pure state: experiment however, shewed, that none of the peculiar products were derived from it; for when mixed with half its weight of sulphur and distilled, mere traces of sulphuretted hydrogen were evolved, and the products were identical with those obtained from the unmixed acid. The nauseous smelling oils being then obviously derived either from the oleic acid, or the glycerine of the oil, I prepared a quantity of pure oleic acid, by the decomposition of the ethereal solution of the oleate of lead. This, when mixed with half its weight of sulphur, and distilled in a capacious retort, underwent decomposition precisely as the crude fixed oil did; sulphuretted hydrogen was developed in great abundance, and the product of the distillation could not be distinguished from that which I had previously obtained. I was unable to obtain glycerine in sufficient quantity to make a separate investigation of the products of its decomposition, but these must also be peculiar, as I could not distinguish the presence of acroleine during any period of the distillation of an oil with sulphur.

The product of the distillation of oleic acid was in the form of a reddish-brown oil, having an extremely nauseous odour, in which that of sulphuretted hydrogen was apparent. When rectified, this sulphuretted hydrogen was driven off, and the first portions which distilled were perfectly transparent and colourless. As the process continued, however, the products became gradually darker in colour, and the last portions which distilled became semisolid on standing, from the deposition of a quantity of white crystalline plates. These were separated by filtration through cloth, expressed strongly, and purified by successive crystallizations from alcohol, until they were entirely free from smell and colour. The product was then in the form of white pearly scales, which possessed acid properties, and were totally insoluble in water; they were not therefore sebacic acid, no trace of which could be discovered among the products; but, on the contrary, possessed all the properties of margaric acid. These crystals were obtained from quantities of oleic acid, prepared at different times, and with the greatest possible care, and must have been formed during the decomposition. In order, however, to set this point at rest, some of the same oleic acid was distilled alone, when abundance of sebacic acid was obtained, and the latter portions of the rectified product did not deposit any crystals on cooling, but remained perfectly fluid. As this solid acid is produced only in comparatively small quantity, and

I was unable to obtain enough of oleic acid, I made use, in preparing it on the large scale, of pure almond oil, which, according to SCHÜBLER and GASSEROW, is entirely free of margarine. The oil which I employed was expressed specially for these experiments, at a temperature slightly above 32°; and in order to satisfy myself of the absence of margaric acid in the products of its ordinary decomposition, a quantity was distilled alone, and the product rectified. The latter portions being collected apart did not deposit margaric acid; and this I have also found to be the case with the ordinary almond oil of commerce, in the expression of which a moderate degree of heat is employed.

In distilling the oil and sulphur on the large scale, it became impossible to perform the process by the simple admixture of the substances, the frothing being so great as inevitably to expel the materials from the retort. After a trial of various methods, I found it most convenient to employ the apparatus, of which this is a sketch. The oil was introduced into a large glass balloon, to the mouth



of which two tubes were adapted, one descending to near the middle, and furnished with a cork at the upper end, the other which constituted the neck of the distilling apparatus passed into a tubulated receiver, kept cold by immersion in water or ice. To the tubulature, a doubly bent tube was affixed, which descended into a vessel of alcohol, for the purpose of retaining any of the more volatile portions which might be carried over by the rapid current of sulphuretted hydrogen. The heat must be applied by means of an open charcoal fire, and the furnace should be so constructed, that the fire may be rapidly withdrawn in the event of the action becoming too violent. It is very desirable too, that the balloon should go down into the furnace, so that it may be entirely surrounded by hot air. The oil is introduced into the balloon, of which it must not occupy more than a fifth, or a fourth at most, along with a few small pieces of sulphur, and heat is gradually applied. So soon as effervescence commences, the cork of the small tube is withdrawn, and a small piece of sulphur is introduced; and this is continued gradually, so as to keep up an uniform action. A dark reddish-brown oil passes

into the receiver, and at the same time sulphuretted hydrogen passes in torrents through the alcohol; it there deposits a certain quantity of oil, and on escaping, may be kept burning during the whole operation, with a flame eight or nine inches high. The principal difficulty of this process consists in regulating the heat, so as to keep up a steady action. If the heat be allowed to fall, the contents of the balloon become so viscid, as inevitably to boil over; and at the same time too high a temperature causes the whole action to go on with excessive violence. I have generally operated on quantities of three pounds, each of which requires a complete day for its distillation, during which time the operator must never leave it, but constantly attend to the regulation of the heat, and the gradual addition of sulphur in small quantities. When a quantity equal to about half the oil employed has distilled over, the remaining mass becomes excessively viscid; and just at this point the balloon frequently cracks, the contents escape, and the whole catches fire, and blazes off with a bright yellow flame, and smell of sulphurous acid.

The product of this distillation, which exactly resembled that of the pure oleic acid, was rectified, and the crystals which deposited from the latter portions were expressed and purified by successive crystallizations in alcohol. They then presented all the characters of margaric acid, and gave the following results of analysis :—

I. { 5·275 grains of the acid gave
14·558 ... carbonic acid, and
5·919 ... water.

II. { 6·358 grains of the acid gave
17·578 ... carbonic acid, and
7·212 ... water.

Which gives the following results per cent.—

	Experiment.		Calculation.		
	I.	II.			
Carbon, . . .	75·27	75·40	75·55	C_{34}	2500·0
Hydrogen, . .	12·51	12·66	12·59	H_{34}	425.0
Oxygen, . . .	12·22	11·94	11·86	O_4	400·0
	100·00	100·00	100·00		3325·0

These results agree completely with the formula for margaric acid, and were farther confirmed by the analysis of its silver salt and ether.

4·643 grains of the silver salt gave 1·325 of silver = 28·53 per cent.
7·926 grains of the silver salt gave 2·284 of silver = 28·70 per cent.
The calculated result for margarate of silver gives 28·65 per cent.

The ether was prepared in the usual manner, by dissolving the acid in absolute alcohol, and passing dry hydrochloric acid gas through the solution. The product, which possessed all the properties of margaric ether, gave the following results of analysis:

5·596	grains of the ether gave	
15·662	...	carbonic acid,
6·399	...	water.

	Experiment.	Calculation.		
Carbon, ...	76·33	76·51	C_{38}	2850·0
Hydrogen, . . .	12·70	12·74	H_{38}	475·0
Oxygen, ...	10·97	10·79	O_4	400·0
	100·00	100·00		3725·0

These analyses establish, in a satisfactory manner, that the acid produced was margaric acid. It is scarcely possible, however, in the present state of the investigation, to give anything like a rational explanation of the mode in which it is here formed. Its production from oleic acid has been already observed by LAURENT as the first product of oxidation by nitric acid; but the action of sulphur is certainly of a very different character, and cannot be considered as bearing any analogy to that of an oxidising agent. The quantity of margaric acid produced does not appear to be constant, but varies with the rapidity of the distillation, and is always most abundant when it is slowly performed.

The oil which distils previous to and along with the margaric acid, and constitutes by far the most abundant product of the action of sulphur upon oleic acid and oil of almonds, is a very complex substance, and contains some of its constituents in very small proportion. On this account I found it necessary to prepare it in very large quantity; and in doing so I abandoned the use of almond oil and employed linseed oil instead, which is a much cheaper substance, and yields the same fluid products. When the product of the action of sulphur is carefully rectified, the first portions which pass over, are perfectly transparent and colourless, highly limpid and mobile, and boil at the temperature of 160° Fahr. Only a small quantity, however, passes at this temperature, and the immersed thermometer gradually rises without indicating any fixed boiling point for the fluid. My first attempts to purify this oil, and separate it into its various constituents, did not afford any satisfactory conclusions. Numerous analyses of the more volatile portions were made without obtaining comparable results, although all indicated the presence of carbon and hydrogen nearly in the proportion of equal atoms. The following are the details of three of these analyses:—

I.	4·657	grains of the most volatile oil gave	
	12·688	...	carbonic acid, and
	5·127	...	water.
II.	5·501	grains of an oil less volatile than the preceding gave	
	15·762	...	carbonic acid, and
	6·292	...	water.
III.	4·191	grains of another portion of oil gave	
	12·185	...	carbonic acid, and
	4·720	...	water.

Which correspond to the following results per cent.:

	I.	II.	III.
Carbon, ...	75·03	78·79	79·95
Hydrogen, . . .	12·20	12·72	12·75

All these oils, when treated with fuming nitric acid, yielded an abundant precipitate of the sulphate of barytes; but as the results of the combustion were not constant, no quantitative determination was made.

The action of precipitants, however, upon this oil, afforded a more satisfactory method of obtaining some of its constituents. It gives, with corrosive sublimate, a bulky white precipitate, and with bichloride of platinum, a yellow compound, the characters of which vary slightly, according as it is prepared from the more or less volatile portion of the oil. Nitrate of silver and acetate of lead, mixed with the alcoholic solution of the oil, produce only a slight cloudiness, but on boiling the solutions, the sulphurets of silver and lead are deposited.

The Mercury Compound. In order to obtain this substance in the pure state, the oil was dissolved in alcohol, and an alcoholic solution of corrosive sublimate added. The precipitate which fell was collected on a filter, and washed with ether, until the oil was thoroughly extracted, for which purpose a considerable quantity of ether is required. It is then boiled with a large quantity of alcohol, which dissolves a part of it, and the solution being filtered hot, allows the compound to deposit, on cooling, in the pure state. It is then in the form of a white crystalline powder, having a very fine pearly lustre, and exhibiting under the microscope crystals of a very peculiar form. They are six-sided tables, two opposite angles of which are rounded off, so as to give them a very close resemblance to the section of a barrel. It possesses, even after long-continued washing with ether, a peculiar slight sickening smell, which becomes more powerful on heating, and its powder irritates the nose. It is insoluble in water, which moistens it with difficulty. It requires several hundred times its weight of boiling alcohol for solution, and is almost entirely deposited, on cooling, in microscopic crystals. In ether, it is almost insoluble. When heated, it is decomposed with the evolu-

tion of a peculiar nauseous smelling oil. The sparing solubility of this compound in alcohol renders its preparation in sufficient quantity for analysis an extremely tedious process, and I have sought in vain for a more abundant solvent. The only substance which I have found capable of taking it up in larger quantity, is coal-tar naphtha, but its employment is inadmissible, as the best which can be procured is an extremely impure substance, and the crystals of the compound deposited from it always acquire a rose or violet tint from some of its impurities. Oil of turpentine likewise dissolves it, but not more abundantly than alcohol.

By many successive solutions in alcohol, I obtained enough of this substance for an analysis, of which the following are the results:—

12·302 grains, dried in vacuo, gave
6·592 ... of carbonic acid, and
3·018 ... of water.

8·061 grains, deflagrated with a mixture of nitre and carbonate of soda, gave 7·297 grains of sulphate of baryta = 1·0067 = 12·48 per cent. of sulphur.

The mercury and chlorine were determined together by mixing the substance with quicklime, and introducing the mixture into a combustion tube. The end was then drawn out into an elongated bulb, into which the mercury sublimed, and which was afterwards cut off, dried in the water-bath, and weighed, both with and without the mercury; the chlorine was determined in the usual way from the residue in the tube.

9·958 grains gave 5·976 mercury = 60·01 per cent., and 4·310 grains chloride of silver = 10·67 per cent. of chlorine.

5·797 grains gave 2·409 of chloride of silver = 10·25 per cent. of chlorine.

These results correspond closely with the formula $C_{16}\ H_{16}\ S_5\ Hg_4\ Cl_2$, as is shewn by the following comparisons:—

	Experiment.		Calculation.		
	I.	II.			
Carbon	14·61	...	14·46	C_{16}	1200·0
Hydrogen	2·72	...	2·42	H_{16}	200·0
Mercury	60·01	...	60·32	Hg_4	5003·6
Chlorine	10·67	10·25	10·67	Cl_2	885·3
Sulphur	12·48	...	12·13	S_5	1005·8
	100·49	...	100·00		8294·7

It is sufficiently obvious that the formula $C_{16}\ H_{16}\ S_5\ Hg_4\ Cl_2$ cannot be supposed to represent the rational formula of this substance. On the contrary, the remarkable analogy between its properties and those of the mercury compound of sulphuret of allyl appear clearly to indicate a similarity in their chemical constitution,—a similarity which, as we shall afterwards see, is borne out by the properties of the platinum compound. I consider this substance to contain an or-

ganic sulphuret, analogous to sulphuret of allyl, the constitution of which must be represented by the formula $C_8 H_8 S_2$, to which I give the provisional name of Sulphuret of Odmyl (from ὀδμη *odor*), and that the rational formula of the mercury compound is—

$$(C_8 H_8 S_2 + Hg_2 Cl_2) + (C_8 H_8 S_2 + Hg_2 S).$$

On contrasting this with the formula of the allyl compound, which is—

$$(C_6 H_5 Cl + Hg_2 Cl_2) + (C_6 H_5 S + Hg_2 S_2),$$

two important points of difference are apparent, namely, that in the new compound we have the sulphuret, and not the chloride, of the base in union with corrosive sublimate, and the presence of subsulphuret in place of sulphuret of mercury in the second member of the compound. It is even possible to approximate more closely the formulæ of the allyl and odmyl compounds, by assuming the sulphuret of odmyl to be represented by $C_4 H_4 S$; in which case, the mercury compound becomes:—

$$\{3 (C_4 H_4 S) + Hg_2 S_2\} + (C_4 H_4 Cl + Hg_2 Cl).$$

This formula is, however, incompatible with its reactions, as it involves the presence of calomel in the compound. Treatment with caustic potash, however, shews that this is not the case; as it immediately becomes yellow, from the separation of oxide of mercury, while the black suboxide would have been formed had calomel been present.

When a current of sulphuretted hydrogen is passed through the mercury compound suspended in water, it becomes rapidly black, a peculiar smell is observed, along with that of sulphuretted hydrogen, and, by distillation, an oil passes over, which is obtained floating on the surface of the water. It is perfectly transparent and colourless. Its smell is peculiar, and resembles the nauseous odour developed by crushing some umbelliferous plants. When dissolved in alcohol, it gives, with corrosive sublimate, a white precipitate, soluble in hot alcohol, from which it is deposited in crystals precisely similar to those from which it had been originally separated, and, with bichloride of platinum, a yellow precipitate, slightly soluble in hot alcohol and ether. This oil is, in all probability, the sulphuret of odmyl $C_8 H_8 S_2$, but the small quantity in which I have been able to obtain it, has prevented my performing any analysis of it.

The Platinum Compound. When a solution of bichloride of platinum is added to the alcoholic solution of the crude oil, a yellow precipitate makes its appearance, which does not fall immediately, but goes on gradually increasing for some time, precisely as is the case with the allyl compound. The properties of this precipitate are not, however, perfectly constant, but vary according to the portion of the oil employed to yield it. That obtained from the more volatile portion has a fine sulphur-yellow colour, but the less volatile oil gives an orange precipitate. It is

insoluble in water, sparingly soluble in alcohol and ether. When heated it becomes black, an oil is evolved smelling exactly like that obtained from the mercury compound and sulphuret of platinum is left behind, which requires a high temperature to drive off all its sulphur, and leaves metallic platinum as a silver-white mass. When treated with hydrosulphuret of ammonia, it is converted into a brown powder, exactly like that obtained under similar circumstances from allyl.

The analysis of the yellow compound has not hitherto given results of a satisfactory character. I have found the amount of platinum to oscillate between 43·06 and 49·66 per cent. The former of these was obtained from the most volatile oil, the latter from that which boiled between 300° and 400° Fahr., and intermediate results were obtained at intermediate temperatures. The results obtained from the oil which boiled at a high temperature were remarkably constant; thus I have found, in different experiments, 49·00, 49·51 and 49·66 per cent. of platinum, which appear to indicate the presence of some compound of rather sparing volatility. The precipitate obtained from the most volatile oil appears to be that corresponding to the mercury compound which has just been described. Of it I have been able only to perform a very incomplete analysis, which is insufficient to establish its constitution, especially as it is impossible to ascertain whether it is a homogeneous substance. As the results, however, approximate to a formula analogous to that of the mercury compound, I give the details, such as they are.

9·155 grains of the platinum compound gave
7·474 ... carbonic acid, and
3·294 ... water.

5·701 grains gave 2·455 grains of platinum, =43·06 per cent.

These results approximate to a formula similar to that of the mercury compound:—viz.

$$(C_8\ H_8\ S_2 + Pt\ Cl_2) + (C_8\ H_8\ S_2 + Pt\ S).$$

	Experiment.	Calculation.		
Carbon, . . .	22·26	20·83	C_{16}	1200·0
Hydrogen, . .	3·99	3·47	H_{16}	200·0
Platinum, . .	43·06	42·84	Pt_2	2466·6
Chlorine, . .	...	15·38	Cl_2	885·3
Sulphur, . .	...	17·48	S_6	1005·8
	...	100·00		5757·7

The analogy which those substances bear to allyl is exceedingly interesting, as shewing the possibility of forming, by artificial processes, substances similar in constitution to so remarkable a compound, which is not a product of decomposition, but exists ready formed in a variety of different vegetables, where it must

obviously be produced under circumstances very different from the artificial substance; for allyl cannot exist at all at a high temperature, and is entirely decomposed at, or even below, its point of ebullition. Unfortunately, however, the examination of this substance is much complicated by the necessity of examining its compounds in place of itself. Had it been possible to separate it directly from the crude oil, the determination of its constitution and that of its compounds would have presented comparatively little difficulty, and been arrived at with much less labour than that expended upon the imperfect details I have been able to accumulate. Another point worthy of observation, is the total alteration of the products of decomposition of oleic acid produced by the presence of sulphur; no sebacic acid, and, in fact, none of its ordinary products being evolved, although all the substances produced contain carbon and hydrogen in the proportion of equal atoms, just as they exist among the ordinary products,—a circumstance which, taking into consideration the abundant evolution of sulphuretted hydrogen, we certainly should not have anticipated.

The oil which remains after the separation of the mercury compound, likewise contains sulphur as one of its constituents; but I have not yet had time to commence the investigation of this part of the subject. The discussion of it, as well as various other points connected with the compounds already described, I hope to make the subject of a future communication.

XXV.—*Experiments on the Ordinary Refraction of Iceland Spar.* By WILLIAM SWAN, Esq. Communicated by Professor KELLAND.

(Read 19th April 1847.)

According to the theory devised by HUYGENS, to explain the phenomenon of double refraction in Iceland spar, a pencil of light transmitted through that substance is divided into two pencils; the index of refraction for the one being constant, while for the other it varies with the inclination of the transmitted light to the optical axis of the crystal.

Dr WOLLASTON, in 1802, verified the spheroidal form of the wave of light, which HUYGENS had assumed to account for the refraction of the extraordinary pencil, by a careful experimental investigation, conducted by means of his elegant instrument for "examining refractive and dispersive powers by prismatic reflection."* In 1810, MALUS, in his *Théorie de la Double Réfraction*, also demonstrated experimentally the accuracy of the Huygenian law for the extraordinary pencil. I have not had an opportunity of consulting the memoir of MALUS, so as to know the precise nature of his experiments, with reference to the refraction of the ordinary ray; but the object of Dr WOLLASTON's researches was simply to prove the law of extraordinary refraction, and the constancy of the index of refraction for the ordinary ray, is therefore tacitly assumed by him.

More recently, Professor MACCULLAGH of Dublin, in order to account for certain phenomena observed by Sir DAVID BREWSTER, in the reflexion of light from Iceland spar, was led to propose a law of double refraction, according to which the ordinary ray in that substance has a variable index of refraction; and at his request, Sir DAVID BREWSTER made an experiment to ascertain whether the ordinary refraction of Iceland spar is different at different inclinations to the axis. Two prisms were cut out of the same piece of spar, so that in one the transmitted ray was at right angles to the axis, and in the other, it was coincident with it; and both being cemented to a plate of glass, had their surfaces ground and polished together, so as to ensure the equality of their refracting angles. It was then found that the images of a narrow slit, illuminated by homogeneous yellow light, seen through the prisms, were perfectly coincident, which proved that the index of refraction for the ordinary ray was the same in both prisms, "within the limits of the errors of observation."†

* Philosophical Transactions, 1802, pp. 365 and 387.

† See Experiment on the ordinary refraction of Iceland spar, by Sir DAVID BREWSTER.—*Notices and Abstracts of Communications of the British Association*, 1843, p. 7.

Some time ago, Mr WILLIAM NICOL of Edinburgh, whose skill in cutting and polishing Iceland spar is well known, requested me to undertake the examination of the ordinary refraction of several prisms of Iceland spar, with which he had the kindness to entrust me, and which he had cut so that the transmitted light is inclined at various angles to the axis. The refractive power of these prisms was examined by means of an instrument devised by me for facilitating such inquiries, and described in the Transactions of the Royal Scottish Society of Arts for 1844, p. 293.* It will be sufficient here to explain that the prism is mounted in front of the telescope of a theodolite, with plates of sextant glass in accurate contact with its faces. The deviation of the refracted rays is then measured as in FRAUNHOFER's method of determining refractive powers; and the refracting angle of the prism is ascertained by measuring the deviation of rays that have suffered two reflexions at the surfaces of the sextant glasses. The prism being placed in its position of minimum deviation, the index of refraction is ascertained from the formula $\mu = \frac{\sin \frac{1}{2} (\theta + \alpha)}{\sin \frac{1}{2} \theta}$ where θ is the angle of the prism, and α the minimum deviation of the refracted rays.

The theodolite I used in this investigation is made by TROUGHTON. The horizontal limb, measuring 6·5 inches in diameter, is furnished with two verniers reading 20″, and the telescope magnifies twelve times. As I had not the means of observing an object at a greater distance than 40 feet, and as the correction for parallax due to the distance of the prism from the centre of the theodolite could not be ascertained with sufficient accuracy, owing to the difficulty of finding the exact position of the pencil of incident rays, I determined to adopt a method for avoiding this correction.

This consisted partly in mounting the prism over the centre of the theodolite by means of a simple and ingenious arrangement suggested by Mr JOHN ADIE. A rod of well-seasoned mahogany, fitted to the *Ys* of the theodolite, was furnished at one end with temporary *Ys*, placed so as to shift the telescope out from the centre of the instrument; while, at the other, it carried a counterpoise to the weight of the telescope. To this I added stays of wire passing from the ends of the rod to the extremities of the horizontal axis of the theodolite, which were tightened by means of screws so as to prevent any lateral shake. The whole apparatus was mounted on a very firm portable tripod, and was sufficiently stable.

But although the prism, from its position at the centre of the instrument, did not suffer any material displacement on turning round the telescope, it was still desirable to get rid of any remaining uncertainty as to the direction of the incident light. The method I devised for effecting this object, was to use a collimator so as to obtain a beam of sensibly parallel rays, and thus to place

* Also in the Edinburgh New Philosophical Journal, January 1844.

the luminous object I observed, virtually at an infinite distance. Having fitted a pair of cross fibres of silk* in the anterior focus of the object-glass of a telescope, I carefully adjusted it to distinct vision on a star, so that, on moving the eye up and down, its image remained fixed on the wires. The eye-piece being then cautiously removed, the wires were illuminated by a lamp; and the beam of rays issuing from the object-glass having been directed upon the prism, the optical axis of the collimator was made parallel to the horizontal limb of the theodolite by means of adjusting screws. A common oil-lamp was used in ascertaining the angles of the prisms; but when the deviation of the refracted rays was observed, the wires were illuminated with the homogeneous yellow light of a spirit-lamp with a salted wick: and it must be regarded as a remarkable proof of the perfect homogeneity of this light, that the refracted image of a single fibre of silk was always distinctly visible with a good prism.

I shall now give the results of the examination of Mr NICOL'S prisms, to which I shall refer according to the numbers he has attached to them. The angle of the prism was generally determined by four, and the deviation of the refracted rays by six observations.

The prism marked No. 1 is cut out of the crystal, so that in the position of minimum deviation the transmitted rays are parallel to the axis; and Mr NICOL has worked with such accuracy, that the images produced by the ordinary and extraordinary rays coincide almost exactly in this position. The angle of this prism was found to be 60° 8′ 8″, the deviation of the refracted rays 52° 14′ 36″, and consequently $\mu = 1{\cdot}658367$.†

The plane of refraction in the prism No. 2 is perpendicular to the axis. Its angle was found to be 44° 29′ 20″, and the deviation of the refracted rays 33° 17′ 8″. From which $\mu=1{\cdot}658366$.

Two other prisms, No. 3 and No. 4, were also examined, in which the transmitted rays are perpendicular to the axis; but in either case the prism is cut so that the plane of refraction differs from that of No. 2.

For No. 3, the angle of the prism was found to be 59° 36′ 32″, the deviation of the refracted rays 51° 25′ 25″, and $\mu=1{\cdot}658384$.

For No. 4, the angle of the prism was found to be 44° 55′ 24″, the deviation 33° 42′ 34″, and $\mu=1{\cdot}658361$.

In No. 5, the transmitted rays are inclined 45° to the axis; the refracting angle of the prism was found to be 45° 3′ 51″, the deviation of the refracted rays 33° 50′ 58″, and $\mu=1{\cdot}658385$.

* Silk is not the most suitable material for the purpose, owing to its transparency; but I could procure no better at the time.

† I have also examined another prism, No. 1, and have found $\theta=44°\ 23'\ 2''$, $\delta=33°\ 11'\ 0''$, and $\mu=1{\cdot}658362$.

In No. 6, one of the faces is a cleavage plane, and the principal section of the prism is in the same plane with the axis. Therefore, since the cleavage plane is inclined 45° 23′ 25″ to the axis, it follows that the inclination of the transmitted rays in the position of minimum deviation is nearly 66° 51′. The angle of this prism was found to be 44° 28′ 29″, the deviation of the refracted rays 33° 16′ 22″, and $\mu = 1{\cdot}658389$.

These results are combined in the following Table :—

Prism.	Inclination of the plane of incidence, or of the principal section of the prism to the optical axis of the crystal.*	Inclination of the transmitted rays to the optical axis.	Index of refraction for the ordinary ray (μ).	Difference of each result from the mean value of μ.
No. 1.	0°	0°	1·658367	−0·000008
2.	90°	90°	1·658366	−0·000009
3.	0°	90°	1·658384	+0·000009
4.	45°	90°	1·658361	−0·000014
5.	0°	45°	1·658385	+0·000010
6.	0°	66° 51′	1·658389	+0·000014
Mean			1.658375	0·000011

From this summary it will be seen, that the greatest difference between the observed index of refraction of any prism and the mean of the whole results is only ·000014; while the difference of the greatest and least results is less than ·00003. So close an agreement in six essentially different cases, seems to render it very probable that the index of refraction is really constant; and the result of the investigation thus confirms the accuracy of the Huygenian law.

* As the term, principal section, is employed in more than one sense, it may be proper to observe, in order to avoid ambiguity, that I use it to denote a plane perpendicular to both faces of the prism.—See Sir John Herschel's Treatise on *Light*, in the *Encyclopædia Metropolitana*, p. 370, art. 197.

XXVI.—*Observations on the Temperature of the Ground at Trevandrum, in India, from May 1842 to December 1845. By* JOHN CALDECOTT, *Esq., Astronomer to the Rajah of Travancore.* Communicated in a Letter to Professor J. D. FORBES.

(Read 15th March 1847.)

122 PALL MALL, *Feb.* 3, 1847.

MY DEAR SIR,—I have taken the opportunity of a short visit to England to bring with me a complete copy, to the end of 1845, of the Observations of Terrestrial Temperature, made at Trevandrum (of which I have already sent you a partial account), and which I have now the pleasure to forward to you herewith, to be dealt with in any way you may think proper. Should you think them worthy of presentation to the Royal Society of Edinburgh for publication in their Transactions, it will be gratifying to me. The observations made with the 12 feet thermometer are now at an end, in consequence of the fracture of the instrument during a high wind last year; but, after rejecting those made immediately on the introduction of the instruments, say all those of 1842, there will still remain three entire years of trustworthy readings, which I hope may be found to be sufficient for any purpose for which such observations in a tropical region may have been considered desirable.

JNO. CALDECOTT.

Professor FORBES, Edinburgh.

EXPLANATION.

These three thermometers, made by Messrs ADIE and SON of Edinburgh, were put down on May 1, 1842. They are buried at the respective depths of 12 French feet, 6 French feet, and 3 French feet, on *the top* of the Observatory hill, which is composed of the stone called "laterite." The situation is about 200 feet above the sea.

In December 1841 and January 1842, they were suspended in the Observatory, and compared with a standard thermometer by Traughton and Simms, as follows, viz.:—

No. 1, or the 12 feet thermometer, by a comparison made 12 times in 24 hours for 29 days, *i. e.*, by 348 comparisons, requires, in order to reduce it to the standard thermometer, the addition to the readings here given of 2°·133. The greatest deviation from this mean of any single comparison is 0°·54.

No. 2, or the 6 feet thermometer, by the same number of comparisons, requires the addition of 2°·172; the greatest deviation from which is 0°·69.

No. 3, or the 3 feet thermometer, by the same number of comparisons, requires the addition of 2°·922; the greatest deviation from which is 0°·57.

Latitude of Trevandrum Observatory, . . .	8° 30′ 32″ N.
Longitude from Greenwich, ...	5^h 7^m 59^s E.

[It may here be stated, that the original registers furnished by Mr CALDECOTT include four observations, daily (except on Sundays), at 6 A.M., Noon, 6 P.M., and Midnight. The extent of the Tables is, consequently, very considerable; and, in printing them, it was necessary to ascertain whether these six-hourly observations possessed a value proportional to the great additional space which they would have occupied. A careful examination shewed that the variation from one hour to another, in the different thermometers, was due *mainly* to the influence of the heat of the air in expanding the *exposed* part of the column of spirit, and did not at all indicate the course of the *diurnal* curves at different depths; nor were there data given for eliminating this discrepancy. It was, therefore, decided to publish the 6 A.M. observations alone,* as representing fairly the general result, and as being less liable to error from the irregular expansion of the exposed column of spirit by the intense heat of the tropical sun.

It will be observed that all the temperatures in the Register, and the means, are UNCORRECTED for the large index error of the instruments. A *corrected* summary will be given, together with a few deductions, at the conclusion of the paper.—J. D. F.]

* The monthly means, four times a day, are, however, given in a separate Table.

When blanks occur, the liquid is above the Scale.

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain.	
1842.	°	°	°	°	in.	
May 1	The Thermometers were put down this day.				0·1852	Sunday
2	83·80	84·02	83·40	82·58	0·0168	
3	83·91	84·16	83·54	84·40	......	
4	84·03	84·37	83·78	84·00	......	
5	84·11	84·54	83·89	79·71	0·1095	
6	84·22	84·74	84·10	79·79	0·3915	
7	84·24	84·83	84·17	75·97	3·8596	
8					0·9639	Sunday
9	84·90	84·82	83·85	77·33	0·6902	
10	84·44	85·21	83·55	78·41	0·4168	
11	84·54	85·32	83·40	81·05	......	
12	84·57	85·32	83·20	81·77	......	
13	84·67	85·35	83·18	81·22	......	
14	84·70	85·33	83·15	81·50	......	
15						Sunday
16	84·90	85·28	83·11	81·24	......	
17	84·76	85·18	83·19	79·57	......	
18	84·84	85·21	83·38	80·73	0·084	
19	84·78	85·17	83·50	82·10	......	
20	84·90	85·23	83·70	82·69	0·0379	
21	84·88	85·21	83·80	82·87	......	
22						Sunday
23	84·98	85·27	84·22	80·51	......	
24	84·99	85·29	84·43	81·13	0·0505	
25	84·88	85·34	84·52	82·58	......	
26	85·02	85·48	84·70	77·44	0·3620	
27	85·00	85·42	84·59	76·08	2·0667	
28	85·20	85·54	84·70	75·00	0·5393	
29					1·6878	Sunday
30	85·08	85·61	83·66	74·87	2·7737	
31	84·88	85·60	83·20	78·89	0·2736	
Means	84·66	85·11	83·77	80·09	14·5134	
June 1	85·02	85·60	82·75	79·58	0·0758	
2	84·88	85·49	82·50	80·30	0·0210	
3	85·14	85·41	82·34	79·18	0·4671	
4	85·12	85·30	82·30	77·45	0·8712	
5					0·4420	Sunday
6	......	85·01	81·63	76·34	4·1502	
7	......	84·80	81·64	78·49	0·3956	
8	......	84·69	81·30	80·53	0·0168	
9	......	84·60	81·42	80·80	......	
10	......	84·55	81·43	80·97	......	
11	......	84·43	81·29	80·89	......	
12						Sunday
13	......	84·41	81·66	80·49	......	
14	......	84·30	81·60	80·60	0·3072	
15	......	84·30	81·83	80·91	......	
16	......	84·30	81·90	81·37	......	
17	......	84·30	82·10	81·34	0·1136	
18	......	84·33	82·17	81·34	0·0253	
19						Sunday
20	......	84·29	82·20	80·23	0·0982	
21	......	84·30	82·38	78·42	0·8840	
22	85·10	84·30	82·49	80·33	0·0243	
23	......	84·30	82·50	78·47	0·0274	
24	85·00	84·34	82·50	77·17	0·8250	
25	85·00	84·35	82·21	77·56	......	
26						Sunday
27	84·98	84·35	82·10	79·23	......	
28	84·94	84·32	81·95	79·45	......	
29	84·91	84·30	81·90	75·44	0·9890	
30	84·96	84·30	81·87	75·02	0·0136	
Means	......	84·58	82·00	79·32	8·7473	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain.	
1842.	°	°	°	°	in.	
July 1	84·90	84·21	81·62	76·35	......	
2	85·00	84·21	81·62	78·32	......	
3						Sunday
4	84·90	84·10	81·29	78·60	......	
5	84·92	84·07	81·21	80·07	......	
6	84·90	84·01	81·18	78·48	0·3409	
7	84·88	83·89	81·00	75·28	0·7702	
8	84·88	83·82	81·20	76·60	......	
9	84·88	83·80	81·20	78·18	......	
10						Sunday
11	84·82	83·71	81·05	78·04	......	
12	84·04	83·70	80·94	79·57	......	
13	84·83	83·68	80·88	80·45	0·0042	
14	84·82	83·60	80·70	80·21	......	
15	84·80	83·57	80·79	76·81	0·4629	
16	84·76	83·49	80·65	74·37	1·6331	
17					0·2947	Sunday
18	84·72	83·40	80·88	78·16	0·2063	
19	84·74	83·41	80·70	78·77	0·0463	
20	84·80	83·40	80·74	79·85	......	
21	84·78	83·42	80·79	80·44	......	
22	84·67	83·31	80·70	80·42	0·0463	
23	84·66	83·30	80·72	80·25	......	
24						Sunday
25	84·60	83·22	80·80	80·49	......	
26	84·60	83·20	80·86	79·97	0·7197	
27	84·55	83·17	80·90	78·03	0·5262	
28	84·55	83·18	80·88	79·77	0·1179	
29	84·51	83·16	81·00	79·33	0·0084	
30	84·52	83·18	81·02	79·50	0·7745	
31						Sunday
Means	84·76	83·62	80·97	78·73	5·9516	
Aug. 1	84·44	83·15	80·96	77·66	0·2652	
2	84·45	83·17	80·95	78·33	0·0842	
3	84·43	83·12	80·85	77·11	0·0589	
4	84·42	83·16	80·91	78·09	0·0547	
5	84·38	83·11	80·71	78·93	0·0337	
6	84·36	83·05	80·52	77·37	0·0295	
7					0·0463	Sunday
8	84·39	83·04	80·59	77·49	0·2947	
9	84·33	83·00	80·55	78·98	......	
10	84·30	82·98	80·45	79·57	0·0421	
11	84·31	82·92	80·41	78·81	0·4462	
12	84·30	82·91	80·35	79·24	......	
13	84·23	82·85	80·39	73·91	0·4840	
14					0·1979	Sunday
15	84·23	82·80	80·39	76·07	0·1726	
16	84·22	82·80	80·30	74·01	1·0691	
17	84·18	82·79	80·10	74·84	0·0379	
18	84·14	82·65	79·89	75·12	0·3031	
19	84·13	82·67	79·80	75·30	0·2021	
20	84·15	82·61	79·50	77·42	......	
21						Sunday
22	84·16	82·59	79·50	78·67	0·0168	
23	84·10	82·50	79·30	79·16	0·1347	
24	84·11	82·25	79·29	79·42	0·0084	
25	84·08	82·39	79·57	80·17	......	
26	84·08	82·38	79·70	80·24	......	
27	84·08	82·30	79·70	80·17	0·0210	
28					0·3872	Sunday
29	84·10	82·27	80·05	80·17	0·0253	
30	84·19	82·21	80·00	78·61	0·0084	
31	83·90	82·25	80·21	78·41	......	
Means	84·23	82·74	80·18	77·90	4·4240	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain.	
1842.	°	°	°	°	in.	
Sept. 1	83·95	82·30	80·44	77·76	0·0168	
2	83·96	82·31	80·30	79·34	......	
3	83·92	82·35	80·51	79·07	......	
4					......	Sunday
5	83·89	82·35	80·45	79·39	0·0758	
6	83·82	82·40	80·60	78·71	......	
7	83·84	82·47	80·76	79·35	......	
8	83·82	82·49	80·75	80·23	......	
9	83·80	82·51	81·02	80·45	......	
10	83·74	82·55	81·10	80·18	......	
11						Sunday
12	83·75	82·64	81·45	79·19	......	
13	83·77	82·73	81·64	79·45	0·0042	
14	83·78	82·80	81·72	79·18	0·0168	
15	83·76	82·87	81·88	78·67	......	
16	83·72	82·92	81·96	79·02	......	
17	83·74	82·99	82·00	78·93	0·3114	
18						Sunday
19	83·72	83·10	82·10	76·52	0·2063	
20	83·72	83·18	82·10	76·66	0·5261	
21	83·71	83·21	82·10	76·12	0·5262	
22	83·71	83·37	82·00	75·46	0·1010	
23	83·69	83·27	81·85	75·77	1·8604	
24	83·73	83·32	81·74	76·56	1·1744	
25					0·9555	Sunday
26	83·72	83·30	81·28	76·67	0·2105	
27	83·72	83·26	81·11	76·37	1·3258	
28	83·74	83·25	80·85	77·87	0·3157	
29	83·71	83·12	80·78	78·82	0·0969	
30	83·73	83·10	80·70	79·46	......	
Means	83·75	82·85	81·28	78·28	7·7238	
Oct. 1	83·72	83·00	80·55	77·84	......	
2						Sunday
3	83·74	82·91	80·50	79·05	0·0379	
4	83·75	82·81	80·49	79·52	......	
5	83·71	82·81	80·50	79·66	......	
6	83·73	82·80	80·51	79·94	......	
7	83·72	82·79	80·50	80·29	......	
8	83·73	82·75	80·64	80·69	......	
9						Sunday
10	83·69	82·69	80·78	80·02	......	
11	83·72	82·75	80·85	80·07	......	
12	83·75	82·70	80·99	79·75	......	
13	83·70	82·72	81·09	80·32	......	
14	83·68	82·75	81·25	80·29	......	
15	83·65	82·72	81·35	80·44	......	
16						Sunday
17	83·64	82·87	81·70	79·22	0·2273	
18	83·65	82·93	81·90	80·19	......	
19	83·64	83·02	82·05	80·04	0·0337	
20	83·63	83·07	82·20	79·48	0·1894	
21	83·65	83·10	82·09	78·69	0·1979	
22	83·61	83·19	82·31	79·01	......	
23					0·0126	Sunday
24	83·65	83·36	82·39	79·63	......	
25	83·64	83·40	82·30	77·69	0·2021	
26	83·63	83·42	82·30	78·97	......	
27	83·63	83·41	82·23	77·45	0·6776	
28	83·64	83·48	82·14	75·47	2·4075	
29	83·65	83·54	82·06	78·50	0·0631	
30					0·1473	Sunday
31	83·68	83·50	81·70	74·50	1·2964	
Means	83·68	83·04	81·44	79·10	5·4928	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain.	
1842.	°	°	°	°	in.	
Nov. 1	83·68	83·49	81·61	76·11	0·0716	
2	83·67	83·42	81·41	77·78	0·6524	
3	83·66	83·40	81·23	77·64	0·0253	
4	83·67	83·38	81·12	79·01	......	
5	83·70	83·32	80·98	78·46	......	
6					1·0354	Sunday
7	83·70	83·21	80·93	76·49	0·6397	
8	83·69	83·10	80·59	76·57	0·3199	
9	83·67	83·06	80·70	75·28	0·1684	
10	83·69	83·01	80·62	75·42	0·8165	
11	83·72	82·97	80·42	75·86	0·6272	
12	83·70	82·93	80·32	76·50	0·2778	
13					1·3132	Sunday
14	83·70	82·75	79·91	77·27	0·8165	
15	83·66	82·61	79·70	78·05	0·0716	
16	83·68	82·54	79·63	78·94	0·1179	
17	83·66	82·45	79·50	78·41	1·4269	
18	83·62	82·38	79·58	78·10	0·0126	
19	83·64	82·30	79·52	79·22	......	
20						Sunday
21	83·54	82·19	79·79	79·25	......	
22	83·56	82·17	79·90	79·32	......	
23	83·58	82·19	79·85	80·06	......	
24	83·58	82·20	80·09	78·62	0·3956	
25	83·51	82·11	79·91	78·32	......	
26	83·52	82·19	80·25	78·54	0·0168	
27						Sunday
28	83·49	82·20	80·39	79·32	......	
29	83·76	82·21	80·35	78·05	......	
30	83·48	82·27	80·48	76·68	......	
Means	83·62	82·69	80·37	77·82	8·8053	
Dec. 1	83·29	82·19	80·35	76·79	......	
2	83·43	82·29	80·52	78·01	......	
3	83·41	82·31	80·55	78·32	......	
4						Sunday
5	83·36	82·30	80·45	76·38	......	
6	83·41	82·38	80·66	77·30	......	
7	83·39	82·37	80·67	78·87	......	
8	83·38	82·40	80·65	77·86	......	
9	83·34	82·43	80·83	78·36	0·0674	
10	83·36	82·45	80·79	79·17	......	
11						Sunday
12	83·36	82·57	81·15	80·20	0·0295	
13	83·35	82·59	81·19	80·21	0·0631	
14	83·34	82·61	81·25	79·29	......	
15	83·34	82·70	81·44	79·97	......	
16	83·32	82·70	81·40	80·08	0·0042	
17	83·32	82·76	81·63	79·63	......	
18						Sunday
19	83·29	82·80	81·70	78·65	......	
20	83·26	82·90	81·96	78·32	......	
21	83·30	82·91	81·90	78·91	......	
22	83·32	83·05	82·15	79·07	......	
23	83·32	83·11	82·28	78·60	......	
24	83·31	83·15	82·15	79·14	......	
25						Sunday
26	83·27	83·19	82·25	80·02	......	
27	83·35	83·35	82·55	79·91	......	
28	83·36	83·40	82·60	79·68	......	
29	83·40	83·51	82·78	81·07	......	
30	83·35	83·41	82·59	...	......	
31	83·34	83·49	82·70	...	......	
Means	83·34	82·89	81·52	78·96	0·1642	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 6 P.M. to 6 P.M.	
1843.	°	°	°	°	in.	
Jan. 1						Sunday
2	83·35	83·52	82·68	79·77	0·090	
3	83·42	83·80	83·15	78·59	……	
4	83·47	83·87	83·29	79·06	……	
5	83·41	83·77	83·05	79·30	……	
6	83·47	83·96	83·31	79·98	……	
7	83·45	83·99	83·00	76·52	……	
8					0·037	Sunday
9	83·52	84·10	83·15	76·55	0·033	
10	83·56	84·12	83·17	75·79	0·183	
11	83·51	84·01	82·79	77·43	……	
12	83·46	84·17	83·11	79·53	……	
13	83·60	84·18	82·78	79·60	……	
14	83·60	84·15	82·50	80·27	……	
15						Sunday
16	83·46	83·99	83·00	78·25	……	
17	83·61	84·01	82·29	78·61	0·465	
18	83·66	84·05	82·50	79·66	0·287	
19	83·67	84·02	82·51	80·31	……	
20	83·64	83·26	82·76	70·31	……	
21	83·70	84·00	82·51	79·91	0·059	
22						Sunday
23	83·70	83·90	82·21	89·91	……	
24	83·77	83·96	82·50	80·17	……	
25	83·75	83·95	82·50	80·90	……	
26	83·74	83·92	82·54	79·67	……	
27	83·78	83·93	82·52	77·37	……	
28	83·75	83·91	82·37	77·36	……	
29						Sunday
30	83·76	83·92	82·71	78·90	……	
31	83·75	83·91	82·55	79·92	……	
Means	83·61	83·94	82·75	79·05	1·154	
Feb. 1	83·76	83·94	82·80	79·60	……	
2	83·81	84·02	83·07	80·29	……	
3	83·79	83·99	82·85	79·53	……	
4	83·82	84·08	83·20	81·92	0·001	
5						Sunday
6	83·84	84·18	83·15	79·29	0·005	
7	83·80	84·18	83·54	79·27	0·027	
8	83·84	84·23	83·38	79·56	……	
9	83·29	84·19	83·25	80·21	……	
10	83·36	84·20	83·36	81·21	……	
11	83·85	84·37	83·49	81·46	……	
12						Sunday
13	83·84	84·46	83·56	81·11	……	
14	83·90	84·47	83·82	80·52	……	
15	83·85	84·42	83·89	79·56	……	
16	83·91	84·53	84·18	79·24	……	
17	83·90	84·60	84·33	79·53	……	
18	83·92	84·68	84·45	79·24	……	
19						Sunday
20	83·94	84·81	84·70	79·44	……	
21	83·84	84·80	84·70	79·41	……	
22	83·95	84·92	84·86	79·64	……	
23	84·02	84·98	84·96	80·27	……	
24	84·00	85·09	85·00	79·91	……	
25	83·99	85·10	85·00	79·51	……	
26						Sunday
27	84·05	85·26	85·23	81·13	……	
28	84·08	85·24	85·32	81·23	……	
Means	83·85	84·53	84·00	80·09	0·033	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 6 P.M. to 6 P.M.	
1843.	°	°	°	°	in.	
March 1	84·11	85·43	85·42	82·05	……	
2	84·12	85·52	85·49	82·01	……	
3	84·12	85·58	85·67	82·32	……	
4	84·18	85·51	85·50	82·97	……	
5						Sunday
6	84·19	85·74	86·09	82·33	……	
7	84·23	85·85	86·24	81·94	……	
8	84·22	85·90	86·29	82·79	0·082	
9	84·24	85·98	86·42	82·41	……	
10	84·27	86·02	86·48	80·49	0·186	
11	84·29	86·12	86·61	80·55	0·558	
12					0·803	Sunday
13	84·44	86·29	86·73	81·03	……	
14	84·38	86·28	86·61	81·01	……	
15	84·40	86·41	86·50	81·56	……	
16	84·39	86·31	86·10	82·10	……	
17	84·46	86·48	86·22	81·15	……	
18	84·45	86·49	86·17	82·46	……	
19						Sunday
20	84·52	86·50	86·22	84·15	……	
21	84·58	86·54	86·33	82·83	……	
22	84·53	86·51	86·37	83·95	……	
23	84·62	86·55	86·58	83·60	……	
24	84·65	86·62	86·74	82·72	……	
25	84·68	86·63	86·82	82·44	0·092	
26						Sunday
27	84·72	86·73	87·08	83·18	……	
28	84·72	86·81	87·01	83·03	……	
29	84·76	86·74	87·11	83·88	……	
30	84·76	86·90	87·14	83·53	……	
31	84·80	86·92	87·18	82·36	……	
Means	84·43	86·27	86·41	82·36	1·721	
April 1	84·82	86·79	87·25	78·60	0·420	
2					0·122	Sunday
3	84·82	……	87·38	78·86	0·007	
4	84·88		87·35	80·16	……	
5	84·92		87·16	79·71	……	
6	84·95		86·94	82·30	0·071	
7	84·92		86·60	83·72	……	
8	84·90		86·35	83·95	……	
9						Sunday
10	85·00	……	86·29	84·45	……	
11	84·90	……	86·35	84·01	……	
12	85·10	……	86·47	84·29	……	
13	85·00	……	86·49	82·69	……	
14	……	……	86·60	81·45	0·227	
15	……	……	86·75	82·51	0·012	
16					1·476	Sunday
17	……	……	86·81	80·20	0·042	
18	……	……	86·69	77·46	……	
19	……	……	86·60	78·36	1·180	
20	……	……	86·38	77·46	1·544	
21	……	……	86·10	81·49	1·411	
22	……	……	85·71	76·36	1·099	
23					0·134	Sunday
24	……	……	85·05	82·57	0·228	
25	……	……	84·84	81·50	……	
26	……	……	84·65	80·55	1·437	
27	……	86·61	84·51	82·53	0·001	
28	……	86·52	84·49	83·72	0·163	
29	……	86·50	84·50	84·09	……	
30						Sunday
Means	84·92	86·60	86·17	81·58	9·274	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 6 P.M. to 6 P.M.	
	°	°	°	°	in.	
1843. May 1	......	86·30	84·50	84·60	......	
2	......	86·27	84·58	83·04	0·007	
3	......	86·18	84·63	85·39	0·057	
4	......	86·20	84·76	84·82	......	
5	......	86·10	84·80	83·75	......	
6	......	86·12	84·86	85·57	......	
7						Sunday
8	......	86·10	85·17	80·34	0·204	
9	......	86·12	85·30	84·33	......	
10	......	86·12	85·38	83·57	0·014	
11	......	86·15	85·37	80·66	0·172	
12	......	86·17	85·38	83·60	......	
13	......	86·25	85·55	81·80	......	
14						Sunday
15	......	86·32	85·44	81·37	0·299	
16	......	86·26	85·38	78·46	0·122	
17	......	86·21	85 40	78·24	0·563	
18	......	86·35	85·30	81·46	......	
19	......	86·30	85·25	80·32	......	
20	......	86·28	84·99	80·97	......	
21					2·407	Sunday
22	......	86·27	84·50	76·99	2·267	
23	......	86·16	83·98	77·25	1·647	
24	......	86·12	83·45	78·81	1·471	
25	......	85·92	82·81	79·97	......	
26	......	85·78	82·52	76·89	0·261	
27	......	85·62	82·19	77·90	2·343	
28					0·902	Sunday
29	......	85·25	81·17	76·50	1·188	
30	......	85·07	81·41	77·23	1·320	
31	......	84·94	81·22	78·97	0·745	
Means	......	86·03	84·26	80·62	15·989	
June 1	......	84·73	81·05	81·35	0·432	
2	......	84·58	80·90	80·16	0·065	
3	......	84·45	81·01	79·23	0·196	
4					0·042	Sunday
5	......	84·25	81·10	79·22	0·211	
6	......	84·17	81·15	78·74	0·451	
7	......	84·13	81·30	77·62	0·128	
8	......	84·09	81·25	79·18	0·560	
9	......	84·04	81·35	79·60	......	
10	......	83·99	81·30	77·55	0·178	
11						Sunday
12	......	83·89	81·21	78·60	0·162	
13	85·00	83·88	81·24	78·60	0·240	
14	85·00	83·81	81·00	79·60	0·454	
15	84·96	83·77	81·16	76·36	2·322	
16	84·90	83·65	80·85	74·97	1·941	
17	84·89	83·54	80·65	75·55	1·263	
18					1·768	Sunday
19	84·81	83·26	81·14	79·39	0·020	
20	84·85	83·28	81·17	78·31	......	
21	84·75	83·08	79·85	78·05	0·147	
22	84·69	83·02	80·03	78 89	0·275	
23	84·65	82·98	79·91	77·26	0·230	
24	84·84	82·93	79·99	75·56	1·168	
25						Sunday
26	84·59	82·87	79·90	79·17	0·858	
27	84·56	82·90	80·05	78·60	0·038	
28	84·54	82·84	80·02	76·61	1·767	
29	84·48	82·62	79·60	76·72	1·780	
30	84·44	82·54	79·55	76·50	0·236	
Means	84·75	83·59	80·68	78·21	16·932	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 6 P.M. to 6 P.M.	
	°	°	°	°	in.	
1843. July 1	84·40	82·51	79·53	75·30	0·177	
2					0·654	Sunday
3	84·34	82·37	79·32	74·34	0·573	
4	84·31	82·31	79·28	76·16	1·144	
5	84·29	82·25	79·12	76·50	0·298	
6	84·25	82·17	78·89	77·54	0·048	
7	84·22	82·15	79·05	76·31	......	
8	84·15	82·05	78·90	74·33	1·411	
9					1·301	Sunday
10	84·11	81·88	78·79	75·31	0·965	
11	84·09	81·85	78·77	76·60	0·338	
12	84·01	81·71	78·60	77·65	0·241	
13	83·95	81·61	78·50	75·30	0·598	
14	83·91	81·55	78·50	76·42	0·787	
15	83·92	81·55	78·54	75·92	0·342	
16					0·270	Sunday
17	83·81	81·41	78·49	78·58	......	
18	83·82	81·46	78·61	78·44	......	
19	83·80	81·47	78·67	78·19	......	
20	83·71	81·40	78·65	77·01	0·166	
21	83·74	81·40	78·25	78·41	0·282	
22	83·79	81·40	78·58	78·62	0·192	
23					0·080	Sunday
24	83·62	81·38	78·86	78·95	......	
25	83·60	81·40	78·95	78·64	0·041	
26	83·59	81·38	78·85	78·16	......	
27	83·55	81·39	78·90	78·08	0·062	
28	83·52	81·35	78·89	79·33	......	
29	83·51	81·40	78·95	79·57	......	
30					0·465	Sunday
31	83·45	81·35	79·00	79·30	0·464	
Means	83·90	81·70	78·82	77·29	10·899	
Aug. 1	83·41	81·35	78·99	79·05	......	
2	83·39	81·31	78·99	79·55	......	
3	83·39	81·37	79·05	80·00	......	
4	83·44	81·38	79·07	79·91	......	
5	83·43	81·35	79·10	78·70	......	
6					0·003	Sunday
7	83·24	81·35	79·11	79·28	......	
8	83·27	81·42	79·30	79·49	......	
9	83·28	81·41	79·29	79·72	......	
10	83·23	81·44	79·31	78·62	......	
11	83·30	81·40	79·29	78·79	......	
12	83·18	81·48	79·38	79·41	......	
13						Sunday
14	83·19	81·50	79·53	79·53	......	
15	83·17	81·61	79·60	79·69	......	
16	83·13	81·51	79·70	79·85	......	
17	83·13	81·59	79·79	79·37	......	
18	83·10	81·59	79·90	77·11	0·108	
19	83·07	81·60	80·04	76·23	0·505	
20					0·089	Sunday
21	83·06	81·60	80·02	77·31	0·229	
22	83·05	81·70	80·22	78·05	0·101	
23	83·00	81·73	80·20	78·19	0·541	
24	83·04	81·81	80·21	79·85	0·009	
25	83·04	81·88	80·24	80·55	......	
26	83·00	81·89	79·90	80·66	......	
27						Sunday
28	83·01	81·88	80·12	79·82	......	
29	82·95	81·85	80·13	79·09	0·029	
30	82·95	81·85	80·09	77·67	0·084	
31	83·00	81·90	80·21	78·87	0·400	
Means	83·16	81·58	79·66	79·05	2·098	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1843.	°	°	°	°	in.	
Sept. 1	82·99	81·91	80·31	77·79	0·800	
2	82·98	81·91	80·31	76·48	1·872	
3					0·565	Sunday
4	82·95	81·95	80·20	79·80	0·093	
5	82·92	81·99	80·10	79·10	......	
6	82·98	81·98	80·09	79·09	......	
7	82·95	81·91	79·81	79·17	......	
8	82·91	81·88	79·80	79·58	......	
9	82·91	81·85	79·65	79·64	......	
10						Sunday
11	82·91	81·81	79·61	79·60	......	
12	82·94	81·80	79·70	80·22	......	
13	82·91	81·79	79·70	79·83	......	
14	82·92	81·75	79·92	80·18	......	
15	82·87	81·70	79·85	80·40	......	
16	82·90	81·75	79·98	80·12	......	
17						Sunday
18	82·90	81·81	80·30	79·74	......	
19	82·91	81·80	80·27	79·13	......	
20	82·86	81·85	80·56	79·81	......	
21	82·85	81·88	80·61	80·06	......	
22	82·85	81·92	80·88	80·14	......	
23	82·88	82·03	80·90	80·79	......	
24						Sunday
25	82·80	81·97	81·02	80·44	......	
26	82·85	82·19	81·23	80·37	......	
27	82·81	82·22	81·30	79·94	......	
28	82·81	82·28	81·47	79·45	......	
29	82·85	82·40	81·64	79·95	......	
30	82·84	82·40	81·67	79·01	......	
Means	82·90	81·95	80·46	79·54	3·330	
Oct. 1					0·616	Sunday
2	82·82	82·87	81·88	78·23	......	
3	82·81	82·62	81·81	78·39	0·297	
4	82·82	82·68	81·87	78·05	0·046	
5	82·90	82·82	81·90	77·51	0·266	
6	82·89	82·81	81·86	77·32	0·335	
7	82·88	82·84	81·61	78·22	......	
8						Sunday
9	82·85	82·82	81·27	79·22	......	
10	82·91	82·88	81·50	77·19	1·239	
11	82·94	82·87	81·22	76·36	1·689	
12	82·89	82·80	81·06	78·00	......	
13	82·91	82·82	81·01	78·48	......	
14	82·90	82·71	80·75	79·13	......	
15					0·092	Sunday
16	82·91	82·68	80·50	78·89	......	
17	82·98	82·66	80·61	79·78	......	
18	82·95	82·61	80·60	80·30	0·012	
19	82·95	82·56	80·64	80·37	0·019	
20	82·85	82·48	80·55	79·44	......	
21	82·90	82·48	80·61	79·42	2·290	
22					0·150	Sunday
23	83·00	82·49	80·80	76·92	0·135	
24	82·99	82·49	80·81	79·31	0·046	
25	83·00	82·46	80·89	80·59	......	
26	82·98	82·42	80·85	81·00	......	
27	82·95	82·45	80·80	79·10	0·420	
28	83·00	82·50	80·95	78·39	0·598	
29					0·051	Sunday
30	83·10	82·50	80·91	80·14	......	
31	82·96	82·49	81·08	80·54	0·531	
Means	82·95	82·65	81·09	78·86	8·830	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1843.	°	°	°	°	in.	
Nov. 1	82·95	82·48	81·00	80·72	......	
2	83·10	82·90	81·06	80·72	0·010	
3	82·99	82·56	81·10	79·62	......	
4	82·95	82·51	80·99	81·42	0·566	
5						Sunday
6	82·96	82·52	81·16	79·30	0·343	
7	82·95	82·50	81·05	79·92	0·059	
8	82·92	82·52	81·12	81·05	......	
9	82·98	82·60	81·25	81·52	0·352	
10	82·96	82·57	81·25	80·40	0·256	
11	82·96	82·60	81·32	79·99	0·107	
12						Sunday
13	82·98	82·50	81·39	81·01	......	
14	82·91	82·60	81·40	78·02	0·042	
15	82·99	82·68	81·46	77·99	......	
16	82·98	82·68	81·45	80·19	......	
17	82·95	82·68	81·49	80·19	......	
18	82·91	82·67	81·30	79·45	......	
19						Sunday
20	82·90	82·65	81·18	79·39	......	
21	82·95	82·74	81·21	79·80	......	
22	82·95	82·69	81·52	80·22	......	
23	82·96	82·70	81·20	79·75	0·078	
24	82·90	82·65	80·93	79·38	......	
25	82·95	82·68	81·00	78·51	......	
26						Sunday
27	82·95	82·67	81·07	79·40	......	
28	83·01	82·75	81·15	79·38	......	
29	82·95	82·63	81·01	79·77	......	
30	82·90	82·60	80·95	77·99	......	
Means	82·96	82·63	81·19	79·72	1·813	
Dec. 1	82·92	82·61	80·99	76·80	......	
2	82·94	82·60	81·10	78·42	......	
3					1·468	Sunday
4	83·02	82·86	81·25	76·70	3·200	
5	82·97	82·70	81·15	76·90	3·880	
6	82·98	82·71	81·10	75·90	2·480	
7	82·98	82·62	80·20	77·78	......	
8	82·95	82·58	80·12	79·07	0·212	
9	82·97	82·68	80·13	78·79	0·688	
10						Sunday
11	82·91	82·32	79·72	77·50	......	
12	82·94	82·28	79·76	77·27	1·435	
13	82·90	82·10	79·80	78·50	0·037	
14	82·94	82·16	79·82	78·35	......	
15	82·94	82·10	79·85	76·06	......	
16	82·86	82·02	79·92	74·20	......	
17						Sunday
18	82·81	81·95	79·92	78·81	......	
19	82·88	82·00	79·98	76·50	......	
20	82·86	81·98	79·96	78·50	......	
21	82·92	82·05	79·85	79·57	......	
22	82·86	82·02	79·89	80·00	......	
23	82·82	82·08	79·95	77·08	......	
24						Sunday
25	82·84	81·92	79·85	75·12	......	
26	82·89	82·02	79·93	77·57	......	
27	82·80	81·92	80·00	79·70	......	
28	82·80	81·95	80·00	79·32	......	
29	82·82	82·00	80·00	78·70	......	
30	82·80	82·00	80·10	78·30	......	
31						Sunday
Means	82·90	82·24	80·17	77·69	13·400	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1844.	°	°	°	°	in.	
Jan. 1	82·60	82·06	80·10	77·25	......	
2	82·75	82·01	80·32	76·53	......	
3	82·71	82·00	80·45	76·54	......	
4	82·75	82·08	80·45	76·69	......	
5	82·75	82·10	80·53	77·73	......	
6	82·82	82·06	80·57	79·61	......	
7						Sunday
8	82·75	82·24	81·71	80·09	0·360	
9	82·80	82·00	80·42	78·81	......	
10	82·75	82·31	80·90	77·59	......	
11	82·75	82·34	81·04	78·42	......	
12	82·78	82·33	81·15	77·43	......	
13	82·73	82·40	81·12	77·84	......	
14						Sunday
15	82·75	82·54	81·22	79·36	......	
16	82·77	82·60	81·30	78·90	0·078	
17	82·78	82·65	81·30	78·85	......	
18	82·78	82·60	81·20	78·70	......	
19	82·79	82·70	81·39	77·92	......	
20	82·75	82·69	81·38	78·95	......	
21						Sunday
22	82·80	82·80	81·48	79·12	......	
23	82·88	82·82	81·54	78·93	......	
24	82·80	82·92	81·54	79·45	......	
25	82·84	82·92	81·70	80·25	......	
26	82·81	82·90	81·69	79·96	......	
27	82·85	83·02	81·94	79·58	......	
28						Sunday
29	82·90	83·16	82·14	78·42	......	
30	82·88	83·12	82·22	78·40	......	
31	82·86	83·14	82·25	80·46	......	
Means	82·78	82·54	81·19	78·54	0·438	
Feb. 1	82·98	83·21	82·30	80·46	......	
2	82·90	83·31	82·42	81·02	......	
3	82·95	83·40	82·56	81·89	......	
4						Sunday
5	82·92	83·48	82·58	80·22	......	
6	83·02	83·56	82·63	80·99	......	
7	82·98	83·57	82·90	80·26	......	
8	82·98	83·64	83·07	80·19	......	
9	83·02	83·70	83·22	80·75	......	
10	83·04	83·80	83·42	80·95	......	
11						Sunday
12	83·08	83·94	83·66	79·33	......	
13	83·08	83·97	83·65	80·06	......	
14	83·10	84·08	83·77	79·52	......	
15	83·13	84·12	83·86	80·90	......	
16	83·12	84·25	84·18	79·88	......	
17	83·12	84·24	84·02	78·30	......	
18						Sunday
19	83·18	84·40	84·10	77·20	......	
20	83·20	84·41	84·11	78·10	......	
21	83·23	84·52	84·20	79·25	......	
22	83·28	84·52	84·18	80·33	......	
23	83·26	84·63	84·13	79·61	......	
24	83·28	84·67	84·10	79·85	......	
25						Sunday
26	83·37	84·77	84·23	80·35	......	
27	83·40	84·82	84·40	81·31	0·038	
28	83·46	84·82	84·48	81·94	......	
29	83·44	84·86	84·31	82·30	......	
Means	83·14	84·11	83·62	80·13	0·038	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1844.	°	°	°	°	in.	
March 1	83·45	84·90	84·38	81·28	......	
2	83·50	84·92	84·50	82·50	......	
3						Sunday
4	83·54	85·02	84·60	82·33	......	
5	83·51	85·00	84·58	83·11	......	
6	83·61	85·22	84·90	83·51	......	
7	83·64	85·20	85·12	83·68	......	
8	83·66	85·25	85·16	83·52	......	
9	83·65	85·28	85·30	82·96	......	
10						Sunday
11	83·70	85·42	85·65	82·48	......	
12	83·75	85·52	85·80	82·51	......	
13	83·75	85·57	85·88	81·92	......	
14	83·78	85·60	86·00	82·70	......	
15	83·84	85·72	86·18	83·85	......	
16	83·84	85·82	86·28	84·55	......	
17						Sunday
18	83·90	85·97	86·50	84·45	......	
19	83·92	86·05	86·60	84·41	......	
20	83·95	86·16	86·73	84·55	......	
21	83·92	86·24	86·82	85·81	......	
22	84·02	86·32	87·01	84·82	......	
23	84·03	86·38	87·10	83·71	0·610	
24						Sunday
25	84·21	86·57	87·45	83·89	......	
26	84·14	86·68	87·57	84·05	......	
27	84·16	86·72	87·83	85·58	......	
28	84·20	......	87·50	84·22	......	
29	84·21	......	87·25	84·08	......	
30	84·28	......	87·58	83·95	......	
31						Sunday
Means	83·85	85·72	86·16	83·61	0·610	
April 1	84·32	Above	87·61	85·12	......	
2	84·38	the	87·70	84·00	0·058	
3	84·40	Scale.	87·75	84·40	......	
4	84·47	......	87·90	85·07	......	
5	84·48	......	87·90	84·50	......	
6	84·41	......	87·71	85·40	......	
7						Sunday
8	84·58	......	87·90	84·17	......	
9	84·64	......	87·92	81·42	0·245	
10	84·68	......	87·90	85·07	......	
11	84·65	......	87·88	85·27	......	
12	84·71	......	87·94	84·83	......	
13	84·75	......	87·92	84·13	......	
14						Sunday
15	84·86	......	87·79	84·86	......	
16	84·90	......	87·72	85·17	......	
17	84·80	......	87·81	83·18	0·148	
18	84·90	......	87·80	81·88	0·247	
19	84·90	......	87·81	83·93	......	
20	84·88	......	87·79	85·10	......	
21						Sunday
22	......	......	87·65	85·19	......	
23	......	......	87·64	85·77	......	
24	......	......	87·59	85·10	......	
25	......	......	87·61	84·68	......	
26	......	......	87·61	84·42	......	
27	......	......	87·73	83·32	0·037	
28					1·125	Sunday
29	......	......	87·84	84·97	......	
30	......	......	87·82	85·38	......	
Means	84·65	......	87·78	84·53	1·860	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1844.	°	°	°	°	in.	
May 1	Above	Above	87·93	85·65	······	
2	the	the	87·94	85·65	······	
3	Scale.	Scale.	87·88	84·33	0·213	
4	······	······	87·70	85·38	······	
5						Sunday
6	······	······	87·67	84·41	······	
7	······	······	87·62	84·47	······	
8	······	······	87·53	83·25	······	
9	······	······	87·50	82·31	······	
10	······	······	87·54	84·21	······	
11	······	······	87·54	84·02	······	
12						Sunday
13	······	······	87·42	82·16	······	
14	······	······	87·30	84·48	······	
15	······	······	87·36	82·88	0·160	
16	······	······	87·25	83·41	······	
17	······	······	87·22	83·64	0·218	
18	······	······	87·15	80·80	0·552	
19					1·093	Sunday
20	······	······	87·09	80·39	······	
21	······	······	86·94	81·20	0·144	
22	······	······	86·73	80·77	······	
23	······	······	86·48	80·43	0·445	
24	······	······	86·23	79·11	0·400	
25	······	······	86·08	80·97	0·636	
26					0·687	Sunday
27	······	······	85·61	79·73	0·205	
28	······	······	85·40	80·21	0·075	
29	······	······	85·27	78·91	0·020	
30	······	······	85·09	77·60	0·176	
31	······	······	84·93	77·36	1·577	
Means	······	······	86·93	82·11	6·601	
June 1	······	······	84·60	79·11	0·043	
2					0·075	Sunday
3	······	······	84·07	79·52	0·138	
4	······	86·62	83·80	81·62	0·043	
5	······	86·54	83·66	80·05	0·345	
6	······	86·38	83·50	79·22	0·240	
7	······	86·24	83·45	79·50	0·120	
8	······	86·20	83·38	80·24	······	
9					0·159	Sunday
10	······	85·98	83·22	80·49	······	
11	······	85·88	83·10	81·80	······	
12	······	85·81	83·07	80·07	0·130	
13	······	85·72	82·85	79·84	0·043	
14	······	85·62	82·89	77·96	1·434	
15	······	85·50	82·78	76·93	0·811	
16					1·951	Sunday
17	······	85·36	82·48	78·67	0·265	
18	······	85·25	82·27	80·86	······	
19	······	85·12	82·06	81·09	······	
20	······	85·10	82·00	81·40	······	
21	······	84·93	81·82	79·90	0·109	
22	······	84·72	81·80	79·72	0·193	
23					0·055	Sunday
24	······	84·62	81·86	79·57	0·254	
25	······	84·55	81·76	80·09	0·848	
26	······	84·51	81·81	80·54	······	
27	······	84·61	81·88	81·36	······	
28	······	84·45	81·91	81·43	······	
29	······	84·40	81·88	80·52	0·851	
30					0·082	Sunday
Means	······	85·40	82·72	80·06	7·389	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1844.	°	°	°	°	in.	
July 1	······	84·48	81·92	80·19	0·128	
2	······	84·36	81·98	79·98	0·587	
3	······	84·28	82·07	81·21	······	
4	······	84·28	82·05	80·88	······	
5	······	84·23	82·03	80·45	······	
6	······	84·26	82·03	78·26	0·011	
7					0·278	Sunday
8	······	84·17	81·98	75·29	0·335	
9	85·00	84·15	81·90	77·77	0·186	
10	84·98	84·18	81·93	77·65	0·124	
11	84·99	84·16	81·78	77·08	0·580	
12	84·90	84·09	81·78	76·50	0·245	
13	84·90	84·05	81·76	77·67	0·245	
14						Sunday
15	84·97	83·98	81·45	79·45	······	
16	84·88	83·98	81·32	79·87	······	
17	84·90	83·91	81·22	80·38	······	
18	84·90	83·85	81·08	78·80	0·112	
19	84·86	83·75	81·02	77·86	0·055	
20	84·79	83·76	81·08	78·71	0·035	
21						Sunday
22	84·70	83·62	81·03	78·33	0·015	
23	84·68	83·55	81·00	79·43	······	
24	84·67	83·55	81·05	79·53	0·132	
25	84·68	83·52	81·11	76·02	0·834	
26	84·61	83·48	81·03	76·85	0·069	
27	84·60	83·45	81·05	79·54	0·094	
28					0·279	Sunday
29	84·58	83·49	81·10	79·34	······	
30	84·56	83·44	81·12	78·41	0·210	
31	84·50	83·41	81·00	79·09	0·093	
Means	84·78	83·91	81·48	78·69	4·647	
Aug. 1	84·49	83·35	81·00	77·75	0·131	
2	84·50	83·47	81·07	75·64	0·262	
3	84·46	83·39	80·98	76·70	0·169	
4					0·183	Sunday
5	84·39	83·30	80·93	78·25	0·284	
6	84·35	83·20	80·85	79·26	0·312	
7	84·39	83·25	80·78	79·65	······	
8	84·36	83·24	80·62	77·73	1·016	
9	84·35	83·24	80·71	75·85	0·459	
10	84·29	83·10	80·45	77·75	0·481	
11					0·017	Sunday
12	84·29	83·09	80·35	78·77	0·283	
13	84·26	82·98	80·29	78·41	0·147	
14	84·23	82·94	80·22	78·63	0·058	
15	84·28	82·88	80·18	80·07	······	
16	84·18	82·82	80·12	79·82	······	
17	84·22	82·79	80·22	79·30	0·027	
18						Sunday
19	84·15	82·70	80·21	78·37	0·073	
20	84·10	82·60	80·20	78·14	0·030	
21	84·08	82·62	80·37	78·06	0·134	
22	84·10	82·68	80·40	79·20	······	
23	84·02	82·62	80·35	79·67	······	
24	84·00	82·59	80·38	79·64	······	
25						Sunday
26	83·98	82·62	80·50	79·85	······	
27	84·00	82·66	80·51	79·85	······	
28	83·91	82·60	80·61	78·98	······	
29	83·91	82·58	80·50	79·77	······	
30	83·88	82·61	80·67	79·62	······	
31	83·91	82·67	80·71	79·36	······	
Means	84·19	82·91	80·54	78·65	4·066	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1844.	°	°	°	°	in.	
Sept. 1						Sunday
2	83·85	82·66	80·80	77·62	......	
3	83·80	82·65	80·86	76·48	0·264	
4	83·82	82·73	81·03	78·66	......	
5	83·84	82·75	81·12	79·15	......	
6	83·86	82·77	81·16	78·56	0·406	
7	83·80	82·79	81·03	79·10	0·214	
8					0·538	Sunday
9	83·75	82·84	81·13	79·85	......	
10	83·76	82·87	81·12	79·85	......	
11	83·72	82·81	81·00	78·92	......	
12	83·74	82·88	81·01	79·62	......	
13	83·71	82·89	80·78	79·81	......	
14	83·72	82·87	81·02	79·73	......	
15						Sunday
16	83·71	82·87	81·00	79·97	......	
17	83·66	82·85	81·06	80·01	......	
18	83·68	82·81	81·09	79·55	......	
19	83·66	82·88	81·16	80·54	......	
20	83·64	82·82	81·14	79·72	......	
21	83·64	82·90	81·30	80·65	......	
22						Sunday
23	83·64	82·91	81·50	80·48	......	
24	83·66	83·02	81·67	81·20	......	
25	83·62	82·99	81·72	81·50	......	
26	83·62	83·09	81·90	79·53	0·040	
27	83·68	83·08	82·05	80·50	1·060	
28	83·62	83·18	82·29	81·25	......	
29						Sunday
30	83·50	83·26	82·61	80·50	......	
Means	83·71	82·89	81·23	79·70	2·527	
Oct. 1	83·48	83·29	82·59	79·02	0·591	
2	83·58	83·41	82·69	78·69	0·182	
3	83·63	83·50	82·70	80·45	......	
4	83·61	83·50	82·62	78·42	1·288	
5	83·52	83·50	82·58	76·77	1·914	
6					4·203	Sunday
7	83·37	83·34	82·50	76·55	1·043	
8	83·60	83·69	82·10	77·18	1·249	
9	83·55	83·66	81·74	76·12	2·204	
10	83·55	83·52	81·35	76·96	0·416	
11	83·58	83·49	81·00	77·96	0·020	
12	83·56	83·40	80·81	77·26	......	
13					0·014	Sunday
14	83·56	83·20	80·61	79·96	......	
15	83·55	83·10	80·60	79·22	......	
16	83·56	83·07	80·65	80·00	......	
17	83·56	83·01	80·69	80·58	......	
18	83·56	82·98	80·70	79·08	......	
19	83·57	82·88	80·68	80·25	......	
20						Sunday
21	83·58	82·92	80·90	79·59	......	
22	83·58	82·91	80·90	78·21	0·795	
23	83·56	82·90	81·01	80·51	......	
24	83·55	82·87	81·02	79·87	0·286	
25	83·52	82·92	81·11	80·37	......	
26	83·54	82·91	81·12	80·08	0·036	
27					0·042	Sunday
28	83·55	32·95	81·18	80·24	......	
29	83·50	82·95	81·30	78·97	......	
30	83·51	82·97	81·39	78·37	0·146	
31	83·51	83·00	81·40	78·82	......	
Means	83·55	83·18	81·41	78·94	14·399	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1844.	°	°	°	°	in.	
Nov. 1	83·50	82·96	81·51	78·30	0·745	
2	83·50	83·02	81·50	77·70	0·042	
3					0·067	Sunday
4	83·48	83·08	81·62	78·15	0·412	
5	83·54	83·12	81·50	79·06	1·218	
6	83·50	83·11	81·52	79·16	0·249	
7	83·40	83·10	81·45	80·30	......	
8	83·48	83·15	81·40	80·66	......	
9	83·50	83·18	81·40	80·52	0·822	
10						Sunday
11	83·49	83·13	81·38	79·36	0·193	
12	83·49	83·19	81·45	79·25	0·036	
13	83·48	83·12	81·41	80·55	0·025	
14	83·51	83·19	81·50	80·25	......	
15	83·47	83·12	81·49	81·12	......	
16	83·48	83·10	81·41	80·90	0·060	
17						Sunday
18	83·48	83·12	81·45	81·55	......	
19	83·48	83·18	81·82	81·31	......	
20	83·48	83·19	81·99	81·06	......	
21	83·47	83·18	81·94	79·78	......	
22	83·45	83·21	82·04	78·43	......	
23	83·43	83·22	82·09	80·92	......	
24						Sunday
25	83·47	83·33	82·20	79·65	0·004	
26	83·44	83·31	82·20	79·54	......	
27	83·45	83·34	82·24	80·90	......	
28	83·49	83·42	82·29	80·70	......	
29	83·46	83·44	82·21	81·31	......	
30	83·48	83·47	82·25	81·99	......	
Means	83·48	83·19	81·74	80·02	3·873	
Dec. 1						Sunday
2	83·45	83·40	82·39	80·14	......	
3	83·46	83·51	82·30	78·92	......	
4	83·46	83·53	82·40	78·60	......	
5	83·45	83·51	82·45	78·52	......	
6	83·50	83·55	82·52	78·25	......	
7	83·48	83·61	82·50	78·48	0 050	
8					0·256	Sunday
9	83·51	83·70	82·56	78·26	0·014	
10	83·50	83·75	82·58	78·17	......	
11	83·51	83·75	82·50	78·73	......	
12	83·50	83·78	82·41	79·36	......	
13	83·54	83·81	82·50	79·59	......	
14	83·52	83·78	82·38	80·01	......	
15						Sunday
16	83·52	83·72	82·30	80·45	......	
17	83·58	83·88	82·39	79·32	0·230	
18	83·61	83·82	82·42	77·97	1·216	
19	83·50	83·80	82·35	78·77	......	
20	83·59	83·85	82·42	78·77	......	
21	83·60	83·81	82·40	78·86	0·011	
22					0·012	Sunday
23	83·61	83·80	82·32	78·47	......	
24	83·57	83·78	82·20	78·00	......	
25	83·61	83·89	82·21	78·29	......	
26	83·61	83·80	82·12	78·98	......	
27	83·64	83·80	82·12	80·17	......	
28	83·65	83·80	82·15	79·79	0·049	
29					0·058	Sunday
30	83·68	83·78	82·11	80·11	......	
31	83·64	83·79	82·08	78·57	1·804	
Means	83·55	83·73	82·35	79·09	3·700	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1845.	°	°	°	°	in.	
Jan. 1	83·64	83·84	82·14	76·99	0·005	
2	83·65	83·70	82·08	80·23	......	
3	83·65	83·78	82·11	79·50	0·406	
4	83·68	83·77	82·08	79·46	0·705	
5					0·690	Sunday
6	83·66	83·72	82·07	78·33	......	
7	83·65	83·68	82·09	78·55	......	
8	83·59	83·74	82·04	79·16	......	
9	83·66	83·70	82·00	79·36	......	
10	83·69	83·70	82·04	77·98	......	
11	83·68	83·65	81·95	79·17	......	
12						Sunday
13	83·65	83·67	82·11	79·09	......	
14	83·65	83·64	81·80	77·66	......	
15	83·68	83·63	81·61	77·51	......	
16	83·68	83·60	81·55	78·86	......	
17	83·65	83·59	81·60	78·15	......	
18	83·66	83·52	81·38	78·31	......	
19						Sunday
20	83·69	83·56	81·50	77·85	......	
21	83·60	83·50	81·46	79·63	......	
22	83·68	83·47	81·35	79·42	......	
23	83·68	83·50	81·50	79·69	......	
24	83·68	83·55	81·61	80·43	......	
25	83·68	83·48	81·58	82·13	......	
26						Sunday
27	83·66	83·52	81·94	78·97	0·280	
28	83·66	83·52	81·98	79·33	......	
29	83·66	83·57	82·16	81·07	......	
30	83·68	83·60	82·20	80·98	......	
31	83·65	83·59	82·18	80·54	......	
Means	83·66	83·62	81·86	79·20	2·086	
Feb. 1	83·66	83·70	82·40	80·04	......	
2						Sunday
3	83·70	83·77	82·65	80·09	......	
4	83·68	83·78	82·70	80·49	......	
5	83·65	83·81	82·70	80·26	......	
6	83·68	83·90	82·80	80·30	......	
7	83·68	83·92	82·90	80·65	......	
8	83·65	83·90	82·92	81·13	......	
9						Sunday
10	83·64	84·05	83·05	81·47	......	
11	83·76	84·12	83·12	81·08	......	
12	83·70	84·15	83·41	81·04	......	
13	83·70	84·18	83·50	81·78	......	
14	83·72	84·21	83·55	81·30	......	
15	83·71	84·32	83·75	81·00	......	
16						Sunday
17	83·75	84·46	84·10	81·44	......	
18	83·72	84·50	84·20	82·01	......	
19	83·72	84·60	84·32	81·83	......	
20	83·82	84·70	84·50	82·09	......	
21	83·82	84·78	84·61	80·29	......	
22	83·80	84·70	84·69	78·85	0·006	
23					0·960	Sunday
24	83·78	84·95	84·90	80·64	......	
25	83·86	85·08	84·99	80·41	......	
26	83·88	85·10	85·01	81·89	......	
27	83·87	85·15	84·90	81·26	......	
28	83·88	85·18	84·95	81·26	......	
Means	83·74	84·38	83·78	80·94	0·966	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1845.	°	°	°	°	in.	
March 1	83·94	85·28	84·88	82·61	......	
2					0·004	Sunday
3	84·00	85·36	84·87	82·16	......	
4	84·00	85·38	84·80	82·36	......	
5	84·00	85·44	84·99	84·36	......	
6	84·06	85·48	84·90	80·38	0·534	
7	84·06	85·54	85·00	79·38	0·022	
8	84·06	85·45	85·03	82·52	......	
9						Sunday
10	84·11	85·52	85·12	81·43	......	
11	84·13	85·60	85·08	82·43	......	
12	84·17	85·65	85·10	79·23	1·291	
13	84·18	85·65	85·05	79·13	0·263	
14	84·20	85·70	85·05	80·74	0·322	
15	84·24	85·70	85·01	78·89	0·140	
16					0·470	Sunday
17	84·27	85·71	84·84	80·12	0·040	
18	84·28	85·68	84·60	78·58	1·928	
19	84·27	85·68	84·68	80·50	0·286	
20	84·32	85·70	84·42	80·62	......	
21	84·35	85·67	84·38	83·00	......	
22	84·32	85·61	84·20	84·06	......	
23						Sunday
24	84·44	85·61	84·21	84·92	......	
25	84·44	85·58	84·13	84·91	......	
26	84·44	85·53	84·20	85·64	0·003	
27	84·45	85·58	84·25	84·89	......	
28	84·49	85·61	84·40	84·92	......	
29	84·52	85·50	84·49	84·89	......	
30						Sunday
31	84·50	85·51	84·60	85·05	......	
Means	84·24	85·57	84·70	82·22	5·303	
April 1	84·50	85·49	84·70	84·23	0·075	
2	84·49	85·50	84·82	83·90	......	
3	84·52	85·58	84·92	83·50	......	
4	84·60	85·62	85·08	83·84	......	
5	84·55	85·65	85·20	83·76	......	
6						Sunday
7	84·55	85·72	85·31	84·91	......	
8	84·55	85·79	85·40	85·15	......	
9	84·54	85·76	85·48	83·95	......	
10	84·55	85·88	85·54	84·30	......	
11	84·64	85·79	85·61	84·58	......	
12	84·60	86·00	85·78	84·45	......	
13						Sunday
14	84·66	86·12	86·01	84·22	......	
15	84·60	86·15	86·15	84·20	......	
16	84·64	86·13	86·19	84·74	......	
17	84·66	86·20	86·25	84·57	......	
18	84·70	86·37	86·56	83·75	0·280	
19	84·68	86·45	86·50	83·02	0·078	
20					0·275	Sunday
21	84·64	86·50	86·39	82·60	......	
22	84·78	86·65	86·92	83·71	......	
23	84·78	86·69	86·81	84·42	......	
24	84·78	86·70	86·65	81·72	0·162	
25	84·75	86·69	86·70	84·35	......	
26	84·82	......	86·50	83·19	......	
27						Sunday
28	84·84	......	86·41	84·97	......	
29	84·98	......	86·35	83·11	......	
30	84·88	......	86·50	84·83	......	
Means	84·66	86·06	85·72	84·00	0·870	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
	°	°	°	°	in.	
1845.						
May 1	85·00	......	86·50	81·61	0·351	
2	84·92	......	86·60	81·52	1·351	
3	84·98	......	86·64	81·91	0·357	
4					0·595	Sunday
5	84·95	......	86·64	78·92	0·490	
6	84·90	......	86·52	80·81	0·550	
7	84·90	......	86·41	82·67	0·372	
8	......	......	86·45	81·77	0·201	
9	......	......	85·99	79·49	1·211	
10	......	......	85·70	80·48	0·607	
11						Sunday
12	......	......	85·36	82·51	......	
13	......	86·70	85·10	82·90	0·080	
14	......	86·65	85·00	82·63	......	
15	......	86·60	84·80	82·43	......	
16	......	86·50	84·71	82·90	......	
17	......	86·42	84·60	83·27	......	
18						Sunday
19	......	86·22	84·42	81·55	0·880	
20	......	86·25	84·48	81·89	0·230	
21	......	86·17	84·52	82·57	0·042	
22	......	86·10	84·40	83·11	......	
23	......	86·10	84·50	83·78	......	
24	......	86·10	84·50	83·39	......	
25						Sunday
26	......	86·11	84·53	82·83	0·156	
27	......	85·98	84·55	82·79	......	
28	......	85·99	84·47	83·72	......	
29	......	86·00	84·60	83·04	0·021	
30	......	85·92	84·61	79·51	0·320	
31	......	85·90	84·55	78·81	2·158	
Means	84·94	86·22	85·23	82·08	9·902	
June 1					0·087	Sunday
2	......	85·88	84·49	78·95	0·097	
3	......	85·90	84·41	79·10	1·193	
4	......	85·86	84·35	80·77	......	
5	......	85·52	84·32	79·52	0·471	
6	......	85·70	84·09	80·41	0·030	
7	......	85·86	83·91	82·04	0·003	
8						Sunday
9	......	85·72	83·74	80·38	0·922	
10	......	85·58	83·50	79·69	0·022	
11	......	85·52	83·49	78·45	1·649	
12	......	85·45	83·00	80·27	......	
13	......	85·41	83·33	79·94	0·048	
14	......	85·39	83·30	78·09	......	
15					0·655	Sunday
16	......	85·10	82·98	75·99	1·552	
17	......	85·13	82·80	78·23	......	
18	......	85·11	82·71	79·53	0·144	
19	......	85·09	82·73	80·58	......	
20	......	84·91	82·31	81·49	......	
21	......	84·88	82·02	80·07	......	
22					0·047	Sunday
23	......	84·68	81·90	76·75	1·276	
24	......	84·59	81·86	76·03	0·981	
25	......	84·40	81·90	76·36	0·744	
26	85·00	84·20	81·77	76·30	0·355	
27	84·98	84·60	81·69	78·40	0·112	
28	84·92	84·34	81·61	76·43	0·857	
29					1·280	Sunday
30	84·85	84·08	81·33	76·31	1·059	
Means	84·94	85·17	82·95	78·80	13·584	

Date.	No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
	°	°	°	°	in.	
1845.						
July 1	84·86	83·99	80·98	77·11	0·236	
2	84·85	83·84	80·90	80·05	......	
3	84·80	83·70	80·75	80·89	......	
4	84·75	83·70	80·55	80·81	......	
5	84·76	83·50	80·50	80·43	......	
6						Sunday
7	84·72	83·43	80·60	79·57	0·153	
8	84·65	83·37	80·58	79·85	0·087	
9	84·65	83·30	80·60	79·93	0·021	
10	84·60	83·32	80·52	80·82	0·093	
11	84·62	83·25	80·76	80·80	......	
12	84·58	83·20	80·22	78·29	0·140	
13					0·122	Sunday
14	84·50	83·07	80·60	77·21	0·170	
15	84·50	83·15	80·70	74·59	0·182	
16	84·48	83·13	80·75	75·33	0·575	
17	84·43	83·19	80·89	77·51	0·274	
18	84·50	83·05	80·55	78·77	0·502	
19	84·40	83·10	80·50	79·79	0·050	
20					0·003	Sunday
21	84·38	83·00	80·32	80·12	......	
22	84·35	83·00	80·41	78·57	0·160	
23	84·30	82·90	80·40	79·56	0·020	
24	84·26	82·84	80·37	80·21	......	
25	84·25	82·88	80·34	80·06	......	
26	84·40	82·82	80·41	80·17	......	
27						Sunday
28	84·21	82·85	80·61	80·26	......	
29	84·15	82·76	80·62	80·77	......	
30	84·15	82·75	80·60	80·76	......	
31	84·10	82·80	80·78	80·81	......	
Means	84·49	83·18	80·59	79·37	2·788	
Aug. 1	84·10	82·80	80·70	81·08	0·078	
2	84·08	82·78	80·70	81·12	......	
3						Sunday
4	84·05	82·80	81·05	80·76	......	
5	84·00	82·80	81·02	81·34	......	
6	83·90	82·80	81·24	80·90	0·029	
7	83·98	82·86	80·96	81·61	......	
8	83·96	82·80	81·30	81·50	......	
9	83·94	82·92	81·53	80·99	0·043	
10						Sunday
11	83·89	82·87	81·49	80·57	......	
12	83·95	82·89	81·75	81·00	......	
13	83·85	83·05	81·91	81·09	0·149	
14	83·86	83·11	81·90	78·50	......	
15	83·85	83·15	82·00	77·70	0·133	
16	83·80	83·18	82·00	79·90	......	
17						Sunday
18	83·90	83·11	81·90	77·55	0·526	
19	83·76	83·22	81·82	79·28	......	
20	83·81	83·33	81·91	79·70	......	
21	83·79	83·36	81·90	79·77	......	
22	83·80	83·35	81·89	79·75	0·124	
23	83·81	83·34	81·89	80·73	......	
24						Sunday
25	83·76	83·40	81·74	81·31	......	
26	83·88	83·37	81·89	81·23	......	
27	83·79	83·36	81·75	80·23	......	
28	83·75	83·30	81·75	79·68	0·052	
29	83·75	83·32	81·81	78·93	0·005	
30	83·76	83·40	81·98	79·48	......	
31						Sunday
Means	83·89	83·10	81·60	79·27	1·139	

Date.		No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1845.		°	°	°	°	in.	
Sept.	1	83·80	83·39	81·94	79·43	......	
	2	83·75	83·40	82·15	80·00	......	
	3	83·78	83·42	82·10	80·01	......	
	4	83·79	83·51	82·40	79·04	0·030	
	5	83·59	83·55	82·37	78·31	0·104	
	6	83·76	83·55	82·55	78·92	......	
	7					0·025	Sunday
	8	83·70	83·42	82·50	79·70	0·047	
	9	83·70	83·60	82·61	79·47	......	
	10	83·80	83·75	82·65	79·82	......	
	11	83·75	83·72	82·61	79·54	......	
	12	83·70	83·72	82·60	79·52	......	
	13	83·77	83·75	82·62	80·35	......	
	14						Sunday
	15	83·85	83·88	82·87	80·25	......	
	16	83·79	83·85	82·93	80·08	......	
	17	83·80	83·81	82·90	79·57	......	
	18	83·78	83·95	83·20	81·15	......	
	19	83·80	84·00	83·39	80·30	......	
	20	83·76	84·02	83·50	80·89	......	
	21						Sunday
	22	83·80	84·16	83·76	80·67	......	
	23	83·82	84·20	83·80	80·14	......	
	24	83·80	84·15	83·91	79·59	......	
	25	83·80	84·32	83·99	80·45	......	
	26	83·80	84·40	84·10	80·42	......	
	27	83·85	84·48	84·22	81·47	0·012	
	28					0·030	Sunday
	29	83·88	84·61	84·22	81·31	......	
	30	83·85	84·64	84·25	81·32	......	
Means		83·78	83·89	83·08	80·68	0·248	
Oct.	1	83·88	84·69	84·30	80·81	......	
	2	83·88	84·78	84·25	80·10	0·403	
	3	83·92	84·75	84·30	80·61	0·113	
	4	83·98	84·92	84·50	80·61	0·070	
	5						Sunday
	6	83·98	84·91	84·50	79·98	......	
	7	83·95	84·96	84·50	77·35	3·656	
	8	83·95	84·95	84·45	76·79	0·992	
	9	83·98	84·98	84·50	78·42	0·314	
	10	84·00	84·99	84·50	78·14	0·415	
	11	84·00	85·01	84·02	77·32	2·530	
	12					2·771	Sunday
	13	84·00	84·98	84·54	77·04	3·892	
	14	84·05	85·00	83·05	76·94	0·734	
	15	84·00	85·00	82·20	77·58	1·251	
	16	84·16	84·80	81·98	79·50	0·143	
	17	84·14	84·65	81·91	80·60	0·100	
	18	84·15	84·51	81·70	79·35	......	
	19						Sunday
	20	84·16	84·22	81·50	80·05	......	
	21	84·22	84·22	81·50	80·37	......	
	22	84·15	84·12	81·45	80·54	......	
	23	84·21	84·02	81·41	80·10	......	
	24	84·15	83·90	81·32	80·85	......	
	25	84·10	83·88	81·30	80·64	0·074	
	26						Sunday
	27	84·16	83·78	81·34	81·00	......	
	28	84·17	83·75	81·60	80·48	......	
	29	84·13	83·71	81·72	80·08	......	
	30	84·11	83·69	81·80	79·45	......	
	31	84·10	83·73	81·85	79·90	......	
Means		84·06	84·48	82·82	79·43	17·458	

Date.		No. 1. 12 feet Therm. 6 A.M.	No. 2. 6 feet Therm. 6 A.M.	No. 3. 3 feet Therm. 6 A.M.	Mean Temp. of Air.	Rain 8½ A.M. to 8½ A.M.	
1845.		°	°	°	°	in.	
Nov.	1	84·10	83·72	81·87	80·41	......	
	2					1·273	Sunday
	3	84·10	83·72	82·12	79·96	......	
	4	84·10	83·75	82·11	79·87	......	
	5	84·08	83·73	82·20	79·58	......	
	6	84·05	83·75	82·15	81·00	......	
	7	84·10	83·72	82·20	79·22	2·138	
	8	84·05	83·81	82·22	79·41	......	
	9						Sunday
	10	84·00	83·86	82·30	78·79	......	
	11	84·00	83·85	82·25	80·01	......	
	12	84·00	83·85	82·24	79·97	0·015	
	13	84·05	83·90	82·18	76·93	0·420	
	14	84·00	83·90	82·12	79·06	0·281	
	15	84·02	83·89	82·10	80·03	0·007	
	16						Sunday
	17	84·00	83·84	82·05	79·90	......	
	18	83·95	83·78	81·98	77·90	......	
	19	84·00	83·72	81·94	78·43	0·200	
	20	84·00	83·82	81·90	79·33	......	
	21	84·00	83·85	81·81	79·64	0·390	
	22	84·00	83·75	81·80	79·87	0·142	
	23					1·077	Sunday
	24	84·03	83·70	81·84	80·48	......	
	25	83·92	83·74	81·59	79·55	0·386	
	26	83·98	83·70	81·70	79·54	......	
	27	83·96	83·60	81·72	79·49	......	
	28	83·89	83·59	81·70	79·94	......	
	29	83·92	83·57	81·70	79·39	......	
	30						Sunday
Means		84·01	83·36	81·99	79·51	5·422	
Dec.	1	83·92	83·50	81·51	79·22	......	
	2	83·96	83·51	81·75	76·80	2·348	
	3	83·74	83·34	81·33	77·46	0·042	
	4	83·70	83·29	81·30	78·80	......	
	5	83·84	83·42	81·40	79·90	......	
	6	83·92	83·48	81·22	79·99	......	
	7					0·031	Sunday
	8	83·90	83·46	81·23	79·47	0·376	
	9	83·90	83·35	81·15	78·49	0·102	
	10	83·88	83·29	81·10	79·58	......	
	11	83·90	83·50	81·12	79·08	0·040	
	12	83·86	83·25	81·20	79·02	......	
	13	83·86	83·21	81·20	78·91	......	
	14						Sunday
	15	84·00	83·44	81·30	79·44	......	
	16	83·84	83·20	81·19	80·76	......	
	17	83·84	83·20	81·21	80·51	......	
	18	83·82	83·21	81·18	79·97	0·035	
	19	83·80	83·20	81·17	78·81	......	
	20	83·78	83·16	81·20	79·29	......	
	21						Sunday
	22	83·79	83·19	81·29	80·11	0·026	
	23	83·82	83·20	81·10	79·59	......	
	24	83·90	83·18	81·15	80·14	......	
	25	83·75	83·19	81·24	79·85	0·640	
	26	83·76	83·26	81·42	77·60	1·356	
	27	83·80	83·31	81·39	79·54	......	
	28						Sunday
	29	83·75	83·28	81·32	79·84	......	
	30	83·76	83·44	81·52	79·56	......	
	31	83·70	83·30	81·32	79·50	......	
Means		83·80	83·31	81·28	79·31	4·997	

ABSTRACT

Monthly Means of Terrestrial Temperature, at the Trevandrum Observatory.
Lat. 8° 30′ 32″ N. Long. 5° 7′ 59″ E.

	No. 1. 12 feet Thermometer.				No. 2. 6 feet Thermometer.				No. 3. 3 feet Thermometer.				Mean Temp. of Air.	Rain.	Average of Four Daily Observations, corrected for Index Error.		
	A.M. 6.	Noon.	P.M. 6.	Mid.	A.M. 6.	Noon.	P.M. 6.	Mid.	A.M. 6.	Noon.	P.M. 6.	Mid.					
2.																	
y	84·66	84·74	84·73	84·57	85·11	85·25	85·20	85·18	83·77	83·92	83·80	83·79	80·09	14·5134	86·808	87·357	86·742
e	......	......	......	......	84·58	84·66	84·57	84·48	82·00	82·10	82·11	82·01	79·32	8·7473	......	86·742	84·977
y	84·76	84·83	84·87	84·76	83·62	83·61	83·59	83·62	80·97	81·07	81·03	80·99	78·73	5·9516	86·938	85·782	83·901
g.	84·23	84·29	84·24	84·23	82·74	82·82	82·77	82·74	80·18	80·29	80·22	80·21	77·90	4·4240	86·380	84·940	83·148
.	83·75	83·86	83·79	83·74	82·85	82·92	82·88	82·87	81·28	81·38	81·31	81·29	78·28	7·7238	85·918	85·052	84·237
.	83·68	83·78	83·71	83·69	83·04	83·16	83·07	82·99	81·44	81·60	81·51	81·49	79·10	5·4928	85·843	85·237	84·437
v.	83·62	83·72	83·65	83·61	82·69	82·80	82·73	82·69	80·37	80·45	80·38	80·34	77·82	8·8053	85·783	84·899	83·307
.	83·34	83·49	83·41	83·37	82·89	82·98	82·85	82·82	81·52	81·73	81·58	81·51	78·96	0·1642	85·535	85·057	84·507
3.																	
.	83·61	83·74	83·67	83·58	83·94	84·14	84·06	84·02	82·75	82·92	82·86	82·82	79·05	1·154	85·783	86·212	85·759
.	83·85	84·05	83·99	83·92	84·53	84·78	84·70	84·54	81·00	84·30	84·16	84·04	80·09	0·033	86·085	86·809	87·047
r.	84·43	84·60	84·52	84·46	86·27	86·68	86·37	86·31	86·41	86·70	86·55	86·48	82·36	1·721	86·643	88·579	89·457
ril	84·92	85·01	84·94	84·96	86·60	86·68	86·59	86·53	86·17	86·33	86·16	86·11	81·58	9·274	87·090	88·772	89·114
y	......	......	......	......	86·03	86·12	86·07	85·99	84·26	84·38	84·27	84·21	80·62	15·989	......	88·224	87·202
e	84·75	84·78	84·74	84·71	83·59	83·59	83·56	83·53	80·68	80·65	80·60	80·58	78.21	16·932	86·878	85·739	83·549
y	83·90	83·96	83·90	83·88	81·70	81·77	81·70	81·66	78·82	78·91	78·85	78·84	77·29	10·899	86·043	83·879	81·777
g.	83·16	83·22	83·17	83·16	81·58	81·67	81·60	81·58	79·66	79·79	79·74	79·72	79·05	2·098	85·310	83·779	82·649
t.	82·90	82·99	82·97	82·94	81·95	82·06	82·00	81·96	80·46	80·55	80·45	80·44	79·54	3·330	85·083	84·164	83·397
.	82·95	83·01	82·96	82·94	82·65	82·74	82·66	82·65	81·09	81·18	81·11	81·09	78·86	8·830	85·098	84·847	84·039
v.	82·96	83·05	83·00	82·95	82·63	82·75	82·68	82·54	81·19	81·33	81·25	81·20	79·72	1·813	85·123	84·822	84·167
c.	82·90	83·00	82·92	82·89	82·24	82·35	82·26	82·22	80·17	80·26	80·20	80·10	77·69	13·400	85·060	84·439	83·104
44.																	
a.	82·78	82·92	82·84	82·78	82·54	82·73	82·64	82·58	81·19	81·43	81·30	81·20	78·54	0·438	84·963	84·794	84·202
b.	83·14	83·29	83·21	83·17	84·11	84·34	84·23	84·15	83·62	83·90	83·74	83·64	80·13	0·038	85·335	86·379	86·647
ar.	83·85	84·04	83·94	83·89	85·72	85·92	85·80	85·73	86·16	86·51	86·33	86·28	83·61	0·610	86·063	87·964	89·242
pril	84·65	84·77	84·68	84·65	......	......	......	......	87·78	88·06	87·86	87·80	84·53	1·860	86·820	......	90·797
ay	......	......	......	......	......	......	......	......	86·93	87·06	86·94	86·84	82·11	6·601	......	......	89·864
ne	......	......	......	......	85·40	85·49	85·38	85·34	82·72	82·82	82·70	82·65	80·06	7·389	......	87·574	85·644
ly	84·78	84·84	84·79	84·74	83·91	84·01	83·91	83·86	81·48	81·57	81·49	81·44	78·69	4·647	......	86·094	84·417
g.	84·19	84·27	84·21	84·15	82·91	83·01	82·92	82·89	80·54	80·63	80·55	80·50	78·65	4·066	86·338	85·104	83·477
pt.	83·71	83·80	83·74	83·69	82·89	83·03	82·95	82·90	81·23	81·44	81·35	81·29	79·70	2·527	85·868	85·114	84·249
t.	83·55	83·66	83·59	83·53	83·18	83·28	83·22	83·17	81·41	81·52	81·42	81·36	78·94	14·399	85·715	85·384	84·349
ov.	83·48	83·59	83·51	83·47	83·19	83·33	83·24	83·19	81·74	81·92	81·81	81·76	80·02	3·873	85·645	85·409	84·729
ec.	83·55	83·78	83·59	83·54	83·73	83·91	83·80	83·73	82·35	82·53	82·41	82·36	79·09	3·700	85·748	85·964	85·334
845.																	
n.	83·66	83·80	83·71	83·65	83·62	83·79	83·68	83·62	81·86	82·30	81·91	81·85	79·20	2·086	85·838	85·850	84·902
eb.	83·74	83·91	83·82	83·74	84·38	84·79	84·49	84·40	83·78	84·06	83·92	83·84	80·94	0·966	85·933	86·687	86·822
ar.	84·24	84·39	84·28	84·22	85·57	85·74	85·61	85·54	84·70	84·88	84·73	84·68	82·22	5·303	86·415	87·787	87·669
pril	84·66	84·79	84·72	84·66	84·06	86·20	86·13	86·06	85·72	86·20	86·09	86·03	84·00	0·870	86·840	88·284	88·932
ay	84·94	......	84·91	84·84	86·22	86·24	86·22	86·17	85·23	85·40	85·23	85·15	82·08	9·902	......	......	88·174
une	84·94	84·98	84·96	84·92	85·17	85·31	85·15	85·03	82·95	83·04	82·89	82·81	78·80	13·584	......	87·337	85·844
uly	84·49	84·59	84·52	84·46	83·18	83·29	83·18	83·14	80·59	80·75	80·63	80·57	79·37	2·788	86.658	85·369	83·557
ug.	83·89	83·99	83·90	83·85	83·10	83·24	83·17	83·11	81·60	81·77	81·65	81·58	79·27	1·139	86.035	85·327	84·572
ept.	83·78	83·90	83·82	83·76	83·89	84·05	83·97	83·89	83·08	83·28	83·17	83·10	80·68	0·248	85·948	86·122	86·079
ct.	84·06	84·20	84·09	84·03	84·48	84·60	84·49	84·41	82·82	82·96	82·85	82·80	79·43	17·458	86·228	86.667	85·779
ov.	84·01	84·14	84·06	84·00	83·36	83·52	83·42	83·35	81·99	82·15	82·06	82·00	79·51	5·422	86·185	85·584	84·972
ec.	83·80	84·01	83·88	83·84	83·31	83·45	83·30	83·28	81·28	81·44	81·32	81·26	79·31	4·997	86·015	85·507	84·247

Remarks on the Preceding Observations. *By* Professor J. D. FORBES.

Mr CALDECOTT'S observations possess an extraordinary interest from being the first of the kind prosecuted between the tropics, from the great care and extent of the observations, and from the circumstances being altogether comparable with those of observations lately made in Europe. [The depths of the thermometers are the same as those at Brussels, Edinburgh, and Greenwich.]

In conformity with Mr CALDECOTT'S suggestion, I have had the *corrected* means of 1843–4–5 united, so as to give the mean temperature of each month (the observations of 1842 being omitted). The results are given in the following Table. The readings of Nos. 1 and 2 are deficient in some of the months, owing to the liquid having risen above the scale :—

MEAN OF THREE YEARS, 1843—5.

	No. 1. 12 feet Thermometer.	No. 2. 6 feet Thermometer.	No. 3. 3 feet Thermometer.	Air Temperature.
January	85·528	85·618	84·954	78·930
February	85·784	86·625	86·838	80·386
March	86·373	88·110	88·789	82·730
April	86·916	88·527*	89·614	83·370
May	....	88·224†	88·413	81·603
June	86·878†	86·883	85·012	79·023
July	86·537	85·114	83·250	78·450
August	85·894	84·736	83·566	78·990
September	85·633	85·133	84·575	79·973
October	85·680	85·632	84·722	79·076
November	85·651	85·271	84·622	79·750
December	85·607	85·303	84·228	78·030
Means	86·043	86·264	85·715	80·025

* Mean of Two Years only. † Result of 1843 only.

The following conclusions are plainly deducible :—

I. The Temperature of the ground at Trevandrum is from 5° to 6° Fahr. *higher* than that of the air. This result is confirmed by observations on the temperature of springs and wells at Trevandrum, which have been obligingly communicated to me by Major-General CULLEN of the Madras Artillery. These observations are printed in the "Proceedings" of this Society.

II. When the monthly means of the thermometers are projected, so as to shew the curves of annual temperature, they are found to have one great inflection and a smaller one. The principal maximum of the temperature of the AIR occurs about the beginning of April, after which the rainy season sets in, and the annual

curve goes through its extreme range in three months; the principal minimum occurring about the middle of July. The remaining fluctuations are comparatively insignificant, but indicate a slight maximum about the middle of October.

III. The epochs of temperature are retarded with the depth below the surface in the usual manner, and, at the same time, casual fluctuations disappear, and the ranges diminish. At 12 French feet, the principal maximum occurs five weeks later than in the open air, and the range is still at least a degree and a half.

From these facts, it is easy to infer that the phenomena of the propagation of heat into the ground near the equator resemble those of temperate latitudes, though modified in extent and character. Mr CALDECOTT's experiments conclusively establish (as he himself has pointed out) the error of the doctrine of BOUSSINGAULT (at least for the eastern hemisphere), that the annual temperature near the equator remains unchanged at the depth of a foot below the surface in the shade. This mistake it is the more important to correct, because M. POISSON has attempted to confirm his mathematical theories of heat by applying them to this alleged fact.*

Mr CALDECOTT's experiments appear farther to prove a considerable excess of the temperature of the earth above that of the air at Trevandrum. This result is in opposition to the opinion of KUPFFER, which supposes the earth temperature to be *less* than that of the air between the tropics, and that of BOUSSINGAULT, which supposes them to be the same.

The results of Mr CALDECOTT are confirmed in both particulars by Captain NEWBOLD of the Madras Army, in a paper lately published in the London Philosophical Transactions.†

* Théorie de la Chaleur, p. 508.

† For 1845, p. 125.

Laggan
Kilfinnan
LOCH LOCHY
LOCH ARKAIG
Muck
Invermealy
Clunes
Kyle Rose
Glen Fintec
Glen Gloy
GLEN ROY
No I
Dalriach
Loch Spey
RIVER
MOEL DHU
Gap
Hill of Bohuntine
Glen Glaster
No III
Craig Dhu
SUPPOSED MARKINGS OF HIGHER SHELVES
SUPPOSED HIGHER SHELVES
East Moy
West Moy
Kirk
Kirk of Kilmanivaig
High Bridge
Manse
Spean Bridge
SPEAN
Tirendrish
Inverroy
Keppoch
Bridge of Roy
Auchnaderry
Kilnahoinet
Corriehoillie
Catholic Chapel
Chasm
Tullich
Inch
Mounessie
Inver Laire
Laire Burn
Courach
Delta
Torgulbin
Reindingach
Muir Shearlich
Tor Castle
Acnadaul
Hill of Unichan
612 ft. above the Sea
Inverlochy Castle
BEN NEVIS
BEN CHLIANIC
PA
SHOWIN

MAP OF
PART OF LOCHABER
SHOWING THE SHELVES IN THE GLENS.

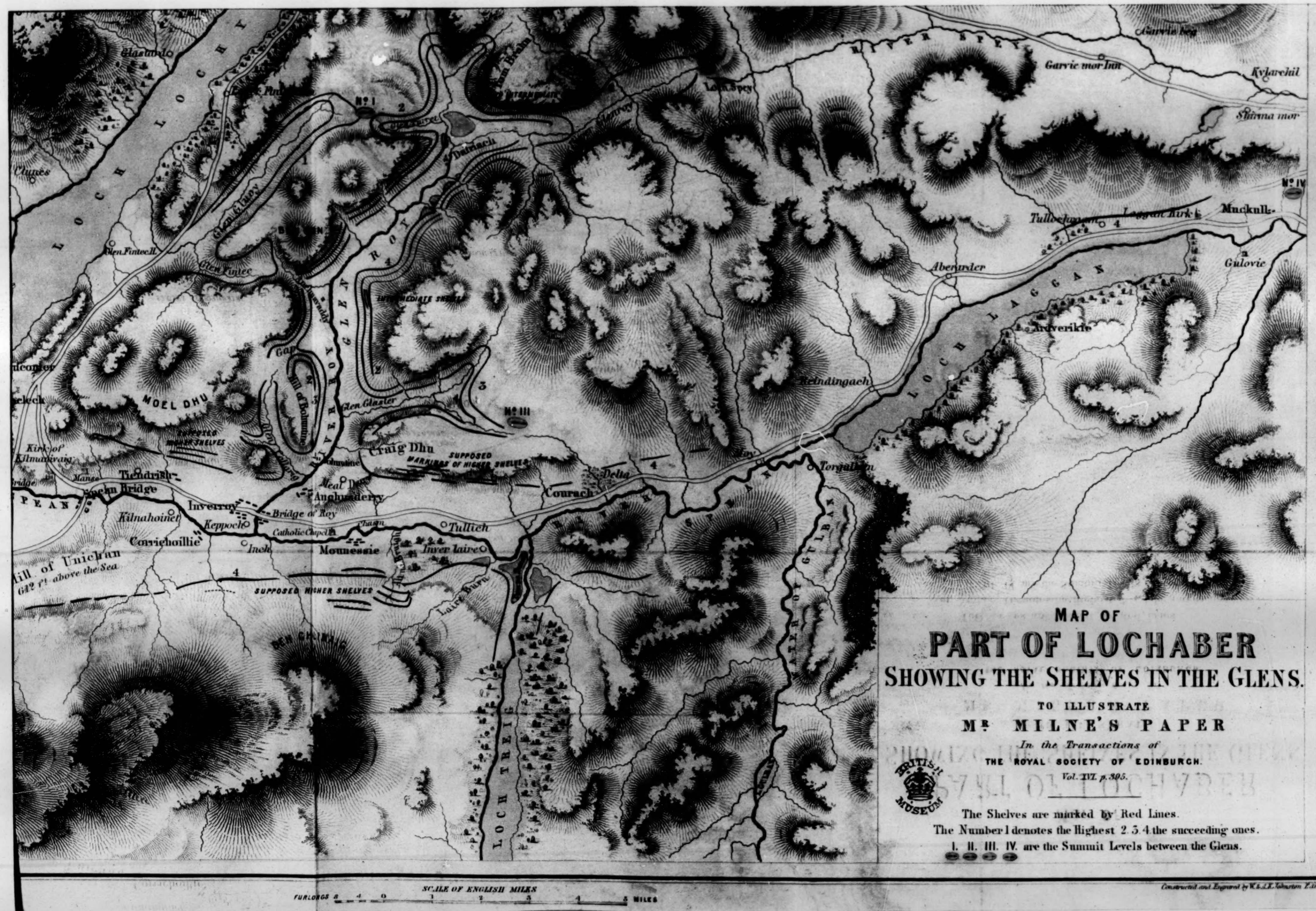
TO ILLUSTRATE

MR MILNE'S PAPER

In the Transactions of

THE ROYAL SOCIETY OF EDINBURGH.

LOCH ARKAIG
LOCH LOCHY
Glen Gloy
Glen Roy
Glen Fintec
MOEL DHU
Hill of Bohuntin
Glen Glaster
Craig Dhu
SUPPOSED MARKINGS OF HIGHER SHELVES
Spean Bridge
Inverroy
Keppoch
Bridge of Roy
Inver Laire
Hill of Unichan
642 ft. above the Sea
SUPPOSED HIGHER SHELVES
BEN NEVIS
FORT WILLIAM
LOCH TREIG
Tor Castle
Muir Shearlich
East Moy
West Moy
Kyle Roy
Clunes
Kilnahoinet
Corrichoillie
Monessie
Tulloch
Delta
Courach
Loch Spey
Reindingach
Torgulbin
Laire Burn
Inverlochy Castle
BRITISH MUSEUM
PA
SHOWIN
SCALE OF ENGLISH MILES
FURLONGS 8 4 0 1 2 3 4 5 MILES


MAP OF
PART OF LOCHABER
SHOWING THE SHELVES IN THE GLENS.
TO ILLUSTRATE
MR MILNE'S PAPER
In the Transactions of
THE ROYAL SOCIETY OF EDINBURGH.
Vol. XVI. p. 395.
The Shelves are marked by Red Lines.
The Number 1 denotes the Highest 2. 3. 4 the succeeding ones.
I. II. III. IV. are the Summit Levels between the Glens.
SCALE OF ENGLISH MILES
FURLONGS 8 4 0 1 2 3 4 5 MILES
LOCH LOCHY
LOCH LAGGAN
LOCH TREIG
RIVER SPEY
GLEN ROY
MOEL DHU
Craig Dhu
SUPPOSED HIGHER SHELVES
SUPPOSED MARKINGS OF HIGHER SHELVES
INTERMEDIATE SHELF
Garvie mor Inn
Kylarchil
Shirma mor
Muckull
Laggan Kirk
Tullochgorm
Aberarder
Gulovie
Ardverikie
Reindingach
Torgulbin
Glen Glaster
Glen Fintec
Clunes
Bridge of Roy
Inverroy
Keppoch
Kilnahoinet
Corrichoillie
Catholic Chapel
Tullich
Inver laire
Mounessie
Inch
Laire Burn
Coanach
Delta
Hill of Unichan
642 ft above the Sea
BEN CHIRAIC
No. I
No. III
No. IV
PEAN

XXVII.—*On the Parallel Roads of Lochaber, with Remarks on the Change of Relative Levels of Sea and Land in Scotland, and on the Detrital Deposits in that Country.* By DAVID MILNE, Esq.

(Read 1st March and 5th April 1847.)

There are few questions in geology which have given rise to so many theories, and so much speculation, as the origin of the parallel roads in the valleys of Lochaber.

In the year 1817, the late Dr MACCULLOCH gave an elaborate description of them, in a paper read before the Geological Society of London. In the year 1818, Sir THOMAS DICK LAUDER read before the Royal Society of Edinburgh a paper, full of equally interesting details. Both of these observers suggested, in explanation of the shelves which mark the mountain sides of these valleys, that they had been occupied by lakes, which, by earthquakes or other violent convulsions, had been drained. This theory was generally received, until, in the year 1839, Mr DARWIN, so justly celebrated as a geologist, and an accurate observer, published his views, and pronounced the shelves to have been formed by the sea; an opinion which, besides being rested on proofs derived from the locality, he enforced also by his observation of similar appearances in South America.

Mr DARWIN's opinion has received the assent of Sir RODERICK I. MURCHISON, Mr LYELL, and Mr HORNER, all successively Presidents of the Geological Society, besides other geologists, both at home and abroad, who are justly regarded as authorities in physical science. Relying on the soundness of their views, I confess that when I went to Glen Roy, in the year 1845, it was with a strong conviction that the lake theory was indefensible; a view to which I was the more inclined, from having studied certain marks along different parts of the Scottish coast, on both sides of the island, which satisfied me that the sea had recently stood at a much higher relative level than at present; and that, in its recession, it had formed, all round our coasts, shelves or beach lines, very analogous to those in the Lochaber valleys. I had not been two days in Glen Roy, before I satisfied myself that these views were inapplicable to the shelves in it and its associated valleys. But I was unable, during my visit of 1845, to remain long enough to obtain evidence of the manner in which the lakes had been dammed up, and eventually drained. I therefore resolved to defer the farther consideration of the subject, until I could pay a second visit. This I accomplished in September 1846, when I spent a week in the examination.

In the following paper, I shall attempt to explain my reasons for thinking

Mr DARWIN'S theory inadmissible, and to point out the manner in which, as it appears to me, that the lakes were drained,—not as supposed by Dr MACCULLOCH and Sir THOMAS DICK LAUDER, by convulsions of nature, but by the gradual operation of ordinary causes.

Though it is the principal object of this paper to account for the formation of the Lochaber shelves, there are no views regarding them which can be suggested, which have not a more general bearing, and the soundness of which may be tested by evidence supplied from other sources. Former writers, accordingly, and especially Mr DARWIN, have felt it to be necessary, after giving their explanation of the parallel roads, to shew, that the principles on which it rests, are, at least, not inconsistent with any established truths in other branches of geology.

I shall not shrink from subjecting the Lake theory, which I have to submit, to a similar ordeal; and the more so, as I feel satisfied that it receives great support from geological considerations now held to be well established.

As the whole details of the parallel roads have been fully described by former writers, I shall limit myself to points on which I have obtained new information, or with regard to which doubts have been expressed.

1. One of the points of the class last referred to, is the absolute horizontality of the shelves. Mr DARWIN, referring to Sir THOMAS DICK LAUDER'S observations on this point (p. 76.), hints at the possibility of errors and omissions in the calculation. M. BRAVAIS, in his paper on the lines of former sea-level in Finmark, suggests, "that an accurate geodetic levelling should be applied in the case of the *doubtful lines* in Scotland," evidently referring to Glen Roy. Mr HORNER, the president of the Geological Society, in his last year's address, observes; "Mr DARWIN'S explanation of the parallel roads of Glen Roy, that they are ancient sea-beaches, appears to be now generally accepted; and it would be most interesting, if it were ascertained by exact levellings, such as those of M. BRAVAIS, *whether they really are parallel.*" Similar doubts had been expressed by Sir R. I. MURCHISON, Mr HORNER'S predecessor, in his anniversary address of 1843; in support of which, he refers to the concurrent opinion of M. de BEAUMONT and Professor PHILLIPS.

In accordance with the doubts expressed by these authorities, the Geological Section of the British Association, at their last meeting, agreed on an address to Her Majesty's Government, requesting them to cause the parallel roads of Lochaber to be examined by the officers of the Ordnance Survey, to ascertain their supposed horizontality.

I have no doubt that the result of this official survey, if made, will be to establish the absolute horizontality of the shelves. In August 1844, Mr D. STEVENSON, at my request, was so obliging as to examine them, and the conclusion at which he arrived, is explained in a letter to me, from which I make the following extracts. "I have had a number of levels taken, the particulars of which I shall

give you afterwards. The *result*, I think, *leaves no doubt as to the perfect horizontality of the 'roads.'* The glen is much more extensive, both as regards length and breadth, than I anticipated, and the height of the roads above its bottom is also very considerable, and any thing like a series of cross sections, referred to the same datum, would be a work of very great magnitude; a month, I should say, would not complete it. The whole we have been able to do, therefore, is to test the uniformity of the levels of the different roads, by viewing them with a good instrument from several points, as was done by Sir THOMAS DICK LAUDER; and, in addition to this, a section was made along the middle road, where it is pretty well defined from Glen Turret downwards, for a distance of nearly $3\frac{1}{2}$ miles, and throughout that stretch, the road was found to be *perfectly horizontal*." . . . "If I had seen that any thing further could be done, I would have left my assistants for a few days longer; they were there a week."

These observations of Mr STEVENSON, whose professional accuracy is undeniable, confirming, as they so completely do, the result of Sir THOMAS DICK LAUDER's survey (and he, too, was aided by an engineer), leave no doubt in my mind, as to the horizontality of the roads. It is scarcely necessary to refer to any farther and weaker testimony on the subject. But it may be proper to add, that during the two occasions when I visited Glen Roy, I had a pocket-level with me, which I constantly used; and that on the last visit I was accompanied by Mr R. CHAMBERS of Edinburgh, who had a larger spirit-level, and we never could detect any deviation from horizontality.

2. There is a point of some importance bearing on the theory of the shelves, about which former observers have disputed. MACCULLOCH found by his barometric observations, that the Glen Gluoy uppermost shelf is 12 feet above the highest in Glen Roy; but he attributed this difference to errors of observation, and his theory in regard to the formation of the shelves proceeds expressly on the assumption, that these shelves are precisely on the same level. Sir THOMAS DICK LAUDER mentions, however, that Mr M'LEAN, the engineer who assisted him, made the Glen Gluoy shelf 12 feet above that in Glen Roy, whilst Sir THOMAS himself made it 15 feet. According to the observations made by myself and Mr CHAMBERS last September, the difference is much greater. By levelling, we made it 29 feet; by joint barometric and sympiesometer observations, I made it 23 feet.

3. Whilst on the subject of Glen Gluoy, I may mention that I discovered in it a second shelf, which the barometer shewed to be 200 feet, and the sympiesometer 213 feet, below the level of the one before referred to. I detected it first immediately above the mouth of Glen Fintec. It is traceable on both sides of the glen, and for several miles upwards.

4. There is a circumstance of great importance, in the theory of these roads, on which I was so fortunate as to obtain farther information. I allude to the fact, that most of the shelves are coincident with some summit level, so as to ad-

mit of the waters flowing over that level as over a lip. Thus the uppermost shelf of Glen Gluoy No. 1, in Sir THOMAS DICK LAUDER's Memoir, is (as he explains) exactly coincident with the water-shed ridge which divides Glen Gluoy from Glen Roy, so that the waters (whatever they were) which stood at that height and formed the beach No. 1, must have flowed out at the head of Glen Gluoy into Glen Roy. In like manner, the uppermost shelf in Glen Roy, No 2 in Sir THOMAS DICK LAUDER's Memoir, is (as he also mentions) exactly coincident with the water-shed ridge which divides Glen Roy from the valley of the Spey; so that the waters which stood in Glen Roy at No. 2 beach, must have flowed over the head of the Glen into Spey valley. In like manner, the only shelf which occurs in Glen Spean, No. 4 in Sir THOMAS DICK LAUDER's Memoir, is exactly coincident with, or rather is a few feet above, the pass of Mukkul at the head of Loch Laggan, through which pass, the waters standing at the level of No. 4 must have flowed eastward into Spey valley. These coincidences, as Mr DARWIN admits, "are so remarkable, that they must (I use his own words) be intimately connected with the origin of the shelves; although such relation is not absolutely necessary, *inasmuch as the middle shelf of Glen Roy, is not on a level with any water-shed.*" (P. 43.)

The middle shelf here alluded to is No. 3 in Sir THOMAS DICK LAUDER's list. The discovery which I made, was its exact coincidence with a water-shed at the head of Glen Glaster, a glen which, though branching up from Glen Roy near the bottom of it, oddly enough does not appear to have been visited, and certainly not to have been described, by any former observer.

Shelves 3 and 4 are the only shelves which enter and run up this glen. Sir THOMAS DICK LAUDER's map inaccurately represents shelf 2 as marking it on both of its sides. Shelf 2 stops, however, on both sides of Glen Roy a little to the eastward of, or above the mouth of Glen Glaster.

In following shelf 3 to the head of this glen, I found that it was there lost in a low mossy flat. A little beyond this flat, and a few feet below the summit-level, an *old river-course* can be distinctly traced down a slope towards Loch Laggan. It has a rocky bed, over which a great body of water had evidently flowed at some former period. The breadth of the rocky bed is from 30 to 40 feet; the knolls of rock are from 2 to 5 feet high, and amongst them are rounded blocks of stone, such as occur in all great Highland rivers. I traced this rocky channel for about a mile towards Loch Laggan; and I afterwards found the place where it had discharged its waters into Loch Laggan, when that loch stood at shelf 4. It is marked by a huge delta, forming a projecting buttress at the level of that shelf, and bulging far beyond the general side of the Laggan valley.

On examining the rocky knolls attentively in this ancient river-course, I found that the smooth faces were all towards Glen Glaster, and the rough faces in the opposite direction, affording proof, if such were needed, that the stream which flowed there had come from Glen Glaster.

A small rivulet trickles now among the rocks, infinitely too feeble to have produced the appearances.

It is now, therefore, established, not only that the whole of the 4 shelves of Lochaber are coincident with water-sheds respectively, but that a great body of water had filled Glen Glaster, and of course Glenroy, the outlet of which was down this ancient river-course to shelf 4 in Loch Laggan, which is at a lower level by 212 feet.

Whilst on this subject, I may mention farther, that I examined narrowly the interval of space between shelf 1 at the head of Glen Gluoy, and shelf 2 at the head of Glen Turret, where the last shelf is nearest to Glen Gluoy. This space also appeared to me to exhibit the features of an ancient river-course, though they are not so striking as those just described. The distance from the one shelf to the other, is about a mile. Where the Glen Gluoy shelf ends, rocky knolls rise above the moss, water-worn below the level of the shelf, but rough above that level. Their smooth faces are all towards Glen Gluoy. Near shelf 2, in Glen Turret, the rocks have evidently been excavated and cut into by some considerable stream; at present a very small burn runs in this rocky channel, quite incapable of producing the appearances.

The grandest exhibition of an ancient and deserted river-course is, however, at the head of Loch Laggan. The Pass of Mukkul is a channel, the bed and sides of which are entirely rock. It is, at its narrowest part, about 70 feet wide, the wall faces being on each side from 40 to 50 feet high. The rocks at the sides are evidently water-worn for about 30 feet up. To the eastward, this gorge expands into a broad channel of several hundred yards in width, divided in the middle by what has formerly been a rocky islet, against which the waters of this large river had chafed in issuing from the pass. For nearly a mile towards the east, the rocky banks continue on each side, but they gradually diverge, having between them a mossy flat sloping gently eastward. The smooth faces of the rocks within the probable reach of the river-waters, are all towards the west, where Loch Laggan is situated. The height of shelf 4 above the highest point of this deserted channel, is, by barometric measurement, about 21 feet, which affords, therefore, some probable estimate of the average depth of the river. I have only to add, that no stream whatever now occupies this water-course, except where, for a short part of it, the river Pattaig flows in a reverse direction into the head of Loch Laggan. This stream was, when I visited it last September, only about 18 inches deep and 30 feet wide, and must be quite inadequate to have formed the rocky banks on each side of it.

The ancient river-course now described is of much greater size than that at the head of Glen Glaster, just as the Glen Glaster river-course is of greater dimensions than those respectively at the head of Glen Gluoy and Glen Roy. The reason is obvious. The river at Mukkul had to discharge not merely the waters

which belonged to Glen Spean, but also those which flowed out from Glen Glaster, comprehending Glen Roy, Glen Collarig, and Glen Gluoy. The Glen Glaster river-course discharged the waters of Glen Collarig, Glen Gluoy, and Glen Roy, whilst the Glen Gluoy stream discharged only the waters of one lake. Mr DARWIN did not visit the Pass of Mukkul. If he had studied the appearances presented by it, and by those almost as strikingly exhibited at Glen Glaster, he would have found it impossible to deny that the waters which formed shelves 3 and 4 flowed down river-courses, and therefore could not be arms of the sea.

His proposition is, "that the waters of the sea, in the form of narrow arms or lochs, such as those now deeply penetrating the western coast, once entered and gradually retired from these several valleys;" and he adds, that after considering the "several and successive steps of the argument, the theory of the marine origin of the parallel roads of Lochaber, appears to me *demonstrated.*" (P. 56.) I regret that Mr DARWIN should have expressed himself in these very decided and confident terms, especially as his survey was incomplete; for I venture to think, that it can be satisfactorily established, that the parallel roads of Lochaber were formed by fresh water lakes.

1. The first circumstance which I shall notice as fatal to Mr DARWIN's theory, is suggested by the fact last referred to, that the waters which formed the different shelves, must have *flowed out of the glens, and descended by river-courses to lower levels.* The waters which formed No. 1 shelf in Glen Gluoy descended nearly 29 feet by flowing into Glen Roy. The waters which formed No. 2 shelf in Glen Roy flowed in like manner into the valley of the Spey. The waters which formed No. 3 shelf were discharged over the head of Glen Glaster, down a slope of about 212 feet in vertical height, into Glen Spean. Lastly, the waters which formed shelf 4 in Glen Spean, issued out of Loch Laggan by the ancient river-course at Mukkul.

Now, any *one* of these cases is irreconcilable with the notion, that the shelves had been formed by arms of the sea. There is no such thing in nature as a river flowing out of an arm of the sea, to a lower level.

Mr DARWIN, as we have seen, admits that this coincidence of the shelves with water-sheds, must be in some way *connected* with their origin; and, accordingly, he endeavours to give an explanation of it consistently with his theory. He says that these water-sheds are *land straits,* with sea on each side of them, and that they consist of littoral deposits or accumulations of matter formed by the opposition of tides. This opinion, however, is altogether inconsistent with the actual circumstances of the case. In the first place, there is at these water-sheds, no accumulation of littoral deposits or detrital matter. They consist, generally, of bared rocks, forming sloping channels or water-courses. In the second place, there is no trace of water at the same level, on each side of these water-sheds. In the third place, when land straits are formed by the accumulation of matter from opposition of tides, it is not in situations like the heads of glens which narrow to a point, and

at that point are separated by a small neck of land,—it is where there is space for a considerable current on each side of the strait.

For these reasons I consider that Mr DARWIN's explanation of the coincidence of the shelves with the water-sheds before described, is quite inadmissible.

2. The second serious objection to Mr DARWIN's theory arises from the fact, that *the shelves* in the different glens *are not coincident in level.* If they had been formed by arms of the sea, as the land rose out of it, the sea should have formed lines in all the valleys which it entered, at precisely the same levels. But neither of the Glen Gluoy shelves is to be seen, in any of the other valleys. So also the No. 2, or highest shelf of Glen Roy, and the next lowest, or No. 3, do not occur in the lower part of that glen, or in the adjoining valleys of Glen Glaster, Glen Spean, and Glen Treig.

Mr DARWIN attempts to explain one, but one only, of these circumstances, viz., the difference of level between No. 1 and No. 2 shelves, by a theory of very questionable soundness. He says, that the tide in Glen Gluoy may have risen 20 feet higher at the head of the estuary, than at the head of Glen Turret. It would be necessary that it should rise 29 feet higher. But if this were the case, then the shelves, at all events, in Glen Gluoy, would not be horizontal, or nearly so:—they would have sloped upwards towards the head of Glen Gluoy, by 29 feet in the course of 6 or 7 miles,—the length of the glen. But this would be inconsistent with the great and well-established fact so characteristic of these Lochaber shelves; and moreover, though the beach-lines at the heads of the two glens might not be exactly coincident in level there, they ought, at all events, to be so at the mouths of the glens where the supposed arms of the sea joined the main body of the ocean,—which is not pretended.

This theory, however, would explain merely the non-appearance of shelf 1 in Glen Roy. The non-appearance of all the others is accounted for by Mr DARWIN, simply by supposing that something or other had prevented them being *marked* in the other glens.

In support of this view, Mr DARWIN refers to two intermediate shelves which are faintly traceable on Tombhran and elsewhere, in order to shew that the water did produce marks at some places, and not at others. But, from the faintness of those intermediate lines, it is manifest that the water had stood at their level for a much shorter period than at the levels of the principal shelves; and, therefore, no fair inference can be drawn from the former applicable to the latter.

3. These considerations suggest, however, a separate and even a more serious objection. Not only should the sea have made markings at the same levels in all the Glens of Lochaber, but it should have produced *similar appearances, and at the same levels respectively, on all the mountains of Scotland*, high enough for the purpose. Mr DARWIN says, "that it would be more proper to consider the *preservation* of these ancient beaches *as the anomaly*, and their obliteration from meteoric agency the ordinary course of nature." (P. 60.) Supposing him right in

this, he ought to have shewn how circumstances caused that anomaly at Glen Roy and its adjoining valleys. But he has not shewn, and cannot shew, that the sides of the Glen Roy mountains, are in any respect different from those other highland mountains. Indeed, he has himself pointed out a similar beach line at Kilfinnin, in a glen towards Inverness. I take leave farther to doubt the soundness of Mr DARWIN's proposition, that the preservation of ancient beach lines is anomalous. The whole of Scotland, and I believe also of the British Islands, is begirt with lines of ancient sea beach.

4. The ancient sea beaches, now alluded to as existing along our coasts, present a very marked contrast with the Lochaber shelves. If these shelves had been formed by the sea, it will, I presume, be admitted that, considering their great altitude, they are of much older date than beach lines at a lower level. *If older, then they should be less perfect and entire.* But the contrary is the case. They are incomparably more perfect and entire than any of the lowest ancient sea terraces which occur along our coasts.

5. If the Lochaber roads were formed by the sea, the well-known actions of the tides, to which Mr DARWIN refers, would have *precluded* the formation of them *along lines absolutely horizontal.*

Mr DARWIN refers to a case in South America, where, in 18 miles, the tidal wave rises at one place 20 feet higher than another in the same estuary. Nearer home, in the Bristol Channel, the sea rises at its head about 50 feet higher than at its mouth.

The tide at Blackwall rises 12 feet higher than at Yarmouth. In the Firth of Tay, the tide rises at Perth 18 inches above the level at Newburgh. The tide at Alloa is said to rise 2 feet 9 inches above its level at Leith. At Glasgow, the tide rises 10 or 11 inches above its level at Greenock. On the Dee, the level of high water is, at Chester, 8 inches above what it is at Flint, near the mouth of the river, a distance of 11 miles.

On this principle, the beaches of Lochaber, if formed by arms of the sea, ought all gradually to rise to the head of the Glens—narrowing, as these glens do, towards the head. But this is negatived by the fact.

6. On more narrowly considering the effect of tidal action, it will readily occur, that the beaches formed by the *sea* must be materially different from those of a *lake*, in which there is no movement of the water at the sides, except such as is caused by winds common to both. In the case of the sea, there is not only a vertical rise and fall of water (which, on the west coast of Scotland, is from 8 to 16 feet) twice in the 24 hours, but also a good deal of lateral current alternately in opposite directions. Hence the *sea*, whilst it will eat into the land more rapidly than a *lake*, will also spread out more completely the detritus washed down into it. In a lake, on the other hand, which has no movements of water either vertical or lateral, the detritus deposited on the sides of a valley occupied by it, will

be scarcely if at all removed, and will thus form projecting buttresses nearly flat in their upper surfaces, and presenting steep escarpments towards the lake.

Now, applying these two principles of tidal action to the shelves of Lochaber, we seek in vain for any actual *indentation* into the sides of the hills. The shelves consist entirely of *buttresses* which stand out from the sides of the mountains; and these buttresses, so far from sloping at an angle little less steep than that of the sides of the mountains (which would be the case with the sea), form flats or terraces which deviate in general very slightly from the horizontal.

7. If the shelves were formed by the action of the sea, *they should be most distinct at places where the hill sides had been most exposed.*

Thus, on the north and north-west sides of Craig Dhu, and on the west side of Bohuntine, where there must, on Mr DARWIN's theory, have been an open expanse of ocean, the shelves should have been most distinct. But at *these places, the three highest shelves are entirely absent;* the fourth alone is visible, though, being the lowest, it must have been less exposed. It is quite anomalous, on the marine theory, that the shelves should not have been formed where the force of waves and of tidal currents must have been greatest, and that they should have been most distinctly formed in the higher and more sheltered parts of Glen Roy.

The hills at the mouth of Glen Roy seem rather to indicate that the highest shelves had not been formed on them,—the very reverse of what might have been anticipated if Mr DARWIN's views are sound. If they had been formed, they would not have been obliterated, as is manifest from the perfect preservation of shelf 4 on Craig Dhu and Bohuntine.

8. Having stated these objections to the theory of Mr DARWIN, I proceed to consider his objections to the theory, that the shelves were formed by lakes.

These objections resolve entirely into the difficulty of explaining the disappearance of the barriers, which must have dammed back the water in the valleys. But it would be no good reason for rejecting an explanation founded on the existence of barriers, even though we could not very clearly account for the disappearance of them, provided that there is direct and conclusive evidence that such barriers existed. Now, I conceive that there is such evidence furnished by the considerations before referred to.

Let us examine, however, the alleged difficulty of explaining, how the waters could have been dammed up in the valleys to the height of the several shelves.

Shelf 2 is distinctly marked on both sides of Glen Roy, down to a certain point,—and also on both sides of Glen Collarig, down to a certain point. At this period, the water flowed from the east end of Glen Roy into the valley of the Spey. Something must have existed, therefore, in both glens at the points above referred to, to prevent the extension of the shelf westward.

Shelf 3, in both glens, extends a little more to the west than shelf 2. We

have seen that, whilst Glen Glaster is exempt from shelf 2, it is well marked on both sides by shelf 3.

To explain these facts, I assume that there was a blockage of some sort, in Glen Roy, which filled the lower part of the valley up to the level of shelf 2, and which blockage extended a little farther east than the mouth of Glen Glaster. I assume also a similar blockage in Glen Collarig, which filled the lower part of the valley, and as far eastward as the place where shelf 2 stops in that glen. This blockage may have been gravel, clay, or any other detrital matter.

Such is the supposed state of things, whilst the waters stood at shelf 2 in Glen Roy; at which period, it will be remembered, they were discharged to the eastward.

Former writers have assumed, that when the waters sunk from shelf 2, the amount of sinking must have been 82 feet, the distance of shelf 3 below shelf 2; and that this sinking had been one act, caused by an earthquake, or other violent operation, which all at once lowered the barrier by that number of feet. But this is a mistake. MacCulloch takes notice of a shelf faintly marked on Tombhran hill, between shelf 2 and shelf 3, though he expresses afterwards some uncertainty about it. In fact, there are two intermediate shelves visible there; and they are also discernible, at precisely the same level on Ben Erin, and also more distinctly near Achavaddy, on the south side of Glen Roy; the one being about 14 feet below shelf 2, and the other about 36 feet lower down.* These two intermediate shelves clearly indicate, that the water which filled the valley, did not all at once sink from shelf 2 to shelf 3. They prove that the water first sunk down 14 feet, and was stationary at this level for some time; that it then sunk down other 36 feet, and continued at this level for some time; and that it again sunk other 32 feet, at which level it remained for a much longer period, till it formed shelf 3.

It is evident, from these facts, that the lowering of the barrier (of whatever material composed) which confined the water in Glen Roy, was a process of a more gradual and ordinary description than what former writers, and especially Mr Darwin, suppose. It is plain, also, that the barrier which kept in the waters was less rapidly worn down, when they stood at shelves 2 and 3, than at either of the intermediate levels. We see that at shelves 2 and 3 the waters flowed over rocky ledges, in the one case into Spey valley, in the other case by Glen Glaster. Is it not fair from this to infer, that at the intermediate shelves, the water flowed over a blockage of such a nature as was capable of being more easily worn down and obliterated, such as detrital matter? It is, at all events, obvious, that when

* There are hummocks or knolls of stratified gravel and sand in Glen Glaster, the tops of which are all about 36 feet above shelf 3. It is probable that they were deposited when the lake stood at one or other of the intermediate points last mentioned.

the water sunk 14 feet, the discharge must have ceased at the east end; and that it henceforward would go on at the west end, probably near the mouth of Glen Glaster. At every other place, the rocky mountain sides rise so high, as to preclude the possibility of overflow or attrition.

Keeping these principles in view, let us suppose that the detrital matter which blocked up the lower parts of Glen Roy extended a very little to the east of the mouth of Glen Glaster. How easy it is to suppose that this detritus was scooped away, so as to allow of the recession of the waters westward, and of their flowing round the east jaw of Glen Glaster, and on towards the head of that glen, from which they would descend to Glen Spean? For this purpose, it is not necessary to suppose, that there was any *lowering* of the supposed barrier in level, even by a single foot. All that is required is the scooping or wearing away of the detritus, so as to allow of the extension of the lake a little to the westward;—a few yards would be sufficient. As the discharge at this first sinking, must have been at the west end, it is fair to infer that the wearing away of detritus took place there; and when once a flow of water was established through detrital matter, the process of removal would go on rapidly, so as to allow of repeated sinkings of the lake, till it reached the water shed at the head of Glen Glaster, the rocky nature of which would for a time stop any farther sinking, and thus allow of the formation of shelf 3.

According to the foregoing views, we see how the waters would, by successive steps, sink from shelf 2 to shelf 3, and, after entering Glen Glaster, form a marking on both of its sides. We see, also, that the same removal of detritus which allowed the formation of shelf 3 in that glen, would allow also the extension of it on Bohantine Hill, beyond the point where shelf 2 terminates.

Whilst this process of attrition was going on in Glen Roy, there need have been no contemporaneous change in the blockage of Glen Collarig. But there also, at some time or other, a similar scooping out of detritus must have taken place, to allow of the extension of shelf 3 beyond the point where shelf 2 terminates.

Nor is it difficult to conceive, how this removal of detritus was effected. Thus, in Glen Collarig, there are, on both sides of the glen, burns of considerable size and power (from the steepness of their channels) which flowed into the lake. There are three of them, which now descend in that part of the glen marked by shelves 2 and 3. If the detritus which formed the blockage in the lower part of the valley consisted of the same loose sand and gravel which now abounds there, forming cliffs from 70 to 80 feet high, nothing is more easy or natural than the scooping of it out, by such means.

The same observations apply to the blockage in Glen Roy, which, to prevent the waters when at shelf 2 flowing into Glen Glaster, must have been near the mouth of Glen Collarig, called Gap in the maps, out of which, from the number of streams in it, a considerable current had flowed.

So far with regard to the first depression to shelf 3, at which period I suppose the Collarig blockage to be still existing (scooped out a little towards the west), and the blockage in Glen Roy to have been, by a similar process, removed below the mouth of Glen Glaster. The next well marked shelf is No. 4, which is seen on Craig Dhu and Bohuntine, and on both sides of Glen Collarig, and which infers the necessity of removing the blockage entirely from both Glen Roy and Collarig.

This may have been, as in the case of the previous depression, a gradual operation. There is no improbability whatever in the ultimate removal by rivers and burns, of a blockage of the nature supposed. There flows into Glen Roy, from Bohuntine hill, and at or near the very place where the blockage must have existed, the Tundrun Burn, the sides of which shew mica-slate rocks cut through by it to the depth of about 70 feet, and detrital matter above these rocks cut through to the depth of 130 feet. If, since the drainage of the lake, it has thus cut through and removed blockage to the depth of 200 feet, of which one-third is solid rock, this rivulet must have had nearly equal power to wash away the more superficial blockage which existed at this place previously to that event.

The same observations apply to the detrital matter in Glen Collarig, which could easily be carried away by the numerous mountain torrents flowing into that glen.

The following is the manner in which Mr DARWIN alleges that the two depressions must have taken place, according to the lake theory. He says, that there are two barriers, one in Glen Collarig, and the other in Glen Roy: "Let one of the two barriers, we will say the smaller one in Glen Collarig, *give way from the effects of an earthquake*, or other cause, the lake will now stand at the level of the middle shelf, the barriers having *given way* 82 *feet vertically*. Again let it burst, and *this* time rather *more than* 212 *feet vertical must be swept away*. Let all this have taken place, but still a barrier nearly a mile long and 800 feet in height is left standing across the mouth of the Roy. Must we suppose that *each time* the barrier in *Glen Collarig* failed, the one in *Glen Roy gave way the same number of feet*, through some strange coincidence?" It is plain, from this representation, that Mr DARWIN had not in his view, the more simple and gradual process of removal which I have ventured to suggest. It is not in the least necessary to imagine, that there was any sudden sweeping away of barriers of the magnitude supposed; and which would certainly imply the existence and operation of some stupendous agent; but the effect of which would, as Mr DARWIN truly says, have also probably obliterated the shelves. The process which I have suggested, implies the continuous working of ordinary and natural agents,—agents which are now seen at this very place, producing results similar to those required.

Mr DARWIN says, that the barrier across the Roy must have been 800 feet high. This is on the assumption, that the valley of the Roy was then of its present depth and form. But is there to be no allowance made, for the removal by the

river Roy of detritus from the valley? It is manifest, from many appearances along its sides, that the river Roy has cut down at least 200 feet below what was the original bottom (whether of lake or estuary), formed when the waters stood at shelf 4; so that the height of the supposed barrier to retain the waters at shelf 2 would not exceed 600 feet above the bottom of the valley, and might be much less, if the valley were more filled up. Mr DARWIN considers it probable (p. 53), that the buttresses existing on the sides of Glen Roy indicate, that the valley, upwards from Bridge of Roy, had been filled with detrital matter to the very level of shelf 4; in which case the blockage or barrier requisite to form a lake at the level of shelf 2, would have been only about 300 feet above the bottom of the valley. My belief, however, is, that the whole not only of the lower part of Glen Roy, but also of the district about Unachan, High Bridge, and Fort-William, was blocked up with detrital matter, which, in the course of time was washed away by rivers; and that, when the blockage of Glen Roy was removed, the depressed waters standing at shelf 4 were dammed back by detrital accumulations near Unachan, so as to force a discharge by the Pass of Mukkul. This 4th, or lowest shelf, seems to me to stretch much farther to the north, on both sides of the Spean, than former observers have noticed. On the hills flanking the east side, this shelf can be traced to within nearly a mile of Spean Bridge. On the opposite side of the valley, it can be traced to within 6 or 7 miles of Fort-William. The width of the valley where this shelf on both sides ceases to be visible is about 4 miles. Across the mouth of this valley, a little beyond a line joining the extreme visible points of shelf 4, lies the high and elongated hill of Tomnempearaichin, the top of which I found, by the level, to be only 50 or 60 feet below shelf 4; and there is no great difficulty in imagining that the whole of this district, as far as Fort-William, where the enclosing hills are greatly higher, was filled by detritus. There are, even now, detrital remnants of enormous size, of which the well-known Hill of Tomnahurich at Inverness (about 180 feet high and half a mile long), and a hill to the west of it (240 feet high), are specimens indicating the prodigious accumulations once existing in the great glen.

To this point I shall revert. But, in the mean time, taking for granted that such detritus did fill the lower parts of the valleys, it is easy to understand how it should have dammed up the waters into lakes, and how, by a gradual and long-continued process of wearing down, this detrital blockage should have been lowered to the requisite extent.

I have endeavoured to explain the damming back and the depressing of the lakes to their successive levels, without imagining that the level of the sea was then different from what it is at present. If the sea stood at a higher level, then the difficulties of the explanation become less; because the valleys must then have been previously less excavated than they now are, by the operation of rivers. There are good reasons for believing, that since the period of the deposit of the

boulder-clay in Scotland, the sea has stood at least 1000 feet higher on the land than at present. Of course, it must have been after the land rose out of the sea to some extent, that the Lochaber shelves could have been formed by lakes; but the lowest of these might have existed when the sea stood 900 feet above its present level, in which case the depth of detrital matter required to dam up the valleys would be comparatively small.

I have attempted to explain how the valleys of Glen Roy, Glen Collarig, and Glen Spean, were blocked up. There still remains Glen Gluoy, which, as before mentioned, contains two shelves, one of which is about 29 feet above the highest of Glen Roy. Glen Gluoy being unconnected with the other valleys, requires a separate blockage. There would be no great difficulty in imagining the existence of detrital blockage in this glen, at the place where its shelves terminate towards the west, as it is generally, throughout its whole course, exceedingly narrow; and being unconnected with Glen Roy (though MacCulloch states the reverse), its blockage may have been worn down at periods, and in a way, independently of Glen Roy and Glen Collarig.

Before, however, forming a very decided opinion as to the position of the blockage applicable to Glen Gluoy, I should like to examine more particularly than I was able to do, some of the other Glens which open into the Caledonian valley on both sides, with the view of ascertaining whether they contain traces of horizontal shelves about the same height. Mr Darwin takes notice of one in the valley of Kilfinnin,* about 10 miles to the eastward, and which he says is (by his barometric observations) about 40 feet above the highest shelf in Glen Roy; in which case it would be only 10 or 11 feet above that in Glen Gluoy, a difference quite within the limits of error.

I have observed several places along the Caledonial Canal, where there are traces of one or more horizontal terraces, at a height of from 650 to 690 feet above the sea. From these considerations, I infer the possibility of there having been a blockage which applied not merely to Glen Gluoy, but to other glens opening into the great Caledonian valley; and it would, therefore, be most important, that future observers should turn their attention to the adjoining districts.

My explanation of the Lochaber shelves depends entirely on the accuracy of the supposition, that the valleys were, in the lower parts of them, filled up with detrital matter, capable of being gradually worn down and washed away. This supposition is not only not improbable on general principles, but is verified to a great extent by the remains of such detrital matter at and above the heights required for it. Thus, in Glen Collarig, there are to be seen, near the east end,

* It is to be regretted that Mr Darwin, when he visited Lochaber, was not provided with a spirit-level His statement as to the horizontality of this shelf at Kilfinnin, depends entirely on ocular inspection and barometric measurements.

and within about half a mile of the place where the blockage must have existed, enormous heaps of boulder-clay, gravel, and sand. These detrital deposits must have existed in Glen Collarig before the shelves were formed, because shelves 2 and 3 are seen distinctly indented upon these deposits; and I was particularly struck with the fact, that these deposits reach to a height of more than 100 feet above shelf 2. Here is proof, that in Glen Collarig, before the formation of the lake which filled it, there was detrital matter of sufficient depth and consistency to have retained water at the required height. At the place where shelf 2 terminates in this glen, the valley, even at present, is only about 236 feet deep, and 300 yards wide, so that the depth of detrital matter does not exceed the limits of probability—nay, is exemplified by the occurrence of much larger accumulations of detritus in all parts of the Highlands.

It is here proper to explain, that there are in these valleys, as elsewhere in Scotland, two distinct sorts of superficial deposits,—the one consisting of the well-known boulder-clay, and the other of ordinary gravel and sand. This boulder-clay exhibits the same general characters, which it commonly possesses elsewhere; it is unstratified, exceedingly obdurate, of a dark-bluish colour, and filled with water-worn boulders. This boulder-clay I found at the following places;—Spean Bridge, where it is covered by sand; Bohuntine Hill, where it is covered with laminated clay, sloping to the centre of the valley, and about 250 feet below shelf 4; Bohina, on the south side of Glen Roy; Inverlair Bridge, near Loch Laggan; Glen Glaster (on the west side of the valley), from 50 to 80 feet *above* shelf 3; Glen Collarig (near the gap), where it rises *above* shelf 2; Glen Gluoy, as seen at the watershed between it and Glen Roy, and *on a level with shelf* 1. The deposit occurs also at Clenichan, at the river Roy, where the mica-slate rocks, through which the river now runs, are covered immediately by boulder-clay,—the boulder-clay being here covered by deposits of irregularly stratified beds of gravel and sand, from 150 to 200 feet thick. At this place, I observed among the boulders in the hill, granites (with red and grey varieties), old conglomerate, and red porphyry,—rocks, all of which must have come from a distance.

From the fact that this boulder-clay occupies alike the highest and lowest parts of the glens; and, more especially, that in several places it is seen distinctly covered over by laminated clay as well as by stratified gravel and sand, it may be inferred that the boulder-clay, with its imbedded blocks, was deposited, certainly not after the drainage of the lakes, but either before the valleys were occupied with water, or during that period.

In regard to gravel and sand, I do not remember having, in Glen Roy or its contiguous valleys, observed any considerable beds of it, so high up as the boulder-clay. But at lower levels, there are everywhere enormous cliffs of it to be seen, several of which I measured, and found to exceed 180 feet in height. These cliffs are formed out of the ancient bottom of the lake or estuary which filled the valleys, and are

composed of materials washed down from higher levels. The adjoining mountains of the district afford ample evidence, that gravel as well as boulder-clay had been, by some cause or other, brought and deposited over all this country, filling the valleys to heights exceeding the highest of the Glen Roy shelves. Thus, on the turnpike road between Tyndrum and Inverournan, near the summit level between the two valleys, which I estimated to be about 1030 feet above the sea, there is great abundance of sand and gravel. On the Black Mount, about 4 miles north of Inverournan, and at a height of 1300 feet above the sea, there is an immense accumulation of gravel and boulders, particularly on the south side of the summit. In the high ground north of Dalwhinnie, which I estimated at 1200 feet above the sea, there are great heaps of gravel, forming mounds and ridges. These facts, taken in connection with the undoubted fact, that detrital matter has been spread over the greater part of Scotland, to a height of at least 1500 feet above the sea, pretty clearly indicate, that detrital matter not only may have been, but actually was spread over the Lochaber district, and filled its several valleys, to the height of at least the highest of the Glen Roy shelves, thus affording ample blockage for its lakes.

I may mention that there are, in this part of the Highlands, several lakes of small size, at very high levels, the existence of which renders the lake theory of the Glen Roy shelves less improbable than to some it may appear. Thus, at the well-known pass of Rest-and-be-Thankful, there is a small lake, which is about 800 feet above the sea, and there are traces of its having stood formerly from 40 to 50 feet higher. To the south and west of Loch Treig about 3 miles, there are two considerable lakes, one called the Lake of Corry, and the other called Benofflap, which appear, from the accounts received of them, to be about 1200 to 1300 feet above the sea. There are several also on the Black Mount, at about the same high level.

Before concluding what I have to say regarding the parallel roads of Lochaber, I may briefly notice the theory, that the lakes which filled them may have been confined by glaciers, or by the moraines of glaciers.

This was one of the districts which, in the opinion of AGASSIZ and BUCKLAND, afforded undeniable proofs of the existence of glaciers. The former published a paper* on the subject, in which he says: "When I visited the parallel roads of Glen Roy with Dr BUCKLAND, we were convinced that the glacial theory alone satisfies all the exigencies of the phenomenon; and as this locality is the best known, I may limit myself to this example for the explanation of all others."

M. AGASSIZ, in the paper now alluded to, explains the grounds on which his theory rests; and it is accompanied by a plan of the locality.

It appears to me, (1.) That the facts on which M. AGASSIZ rests his theory

* Ed. Phil. Journal, vol. xxxiii., p. 236.

are incorrect. (2.) That, assuming as true the facts stated by him, they still afford no evidence that glaciers existed in the Lochaber valleys.

(1.) There are three main facts relied on by M. AGASSIZ. He states, *First*, That in Glen Roy, and in that part of Glen Spean, between Bridge of Roy and Loch Treig, there are 3 shelves visible; *Secondly*, That these shelves all terminate, on both sides of the valley at or near the Bridge of Roy; *Third*, That the bottom of Glen Spean, in front of Loch Treig, is not only polished with that polish characteristic of glaciers, but is, moreover, scratched transversely,—that is to say, at right angles to the direction of the valley, by a cause which evidently proceeded from Loch Treig.

To explain these appearances, it is suggested, that "the supposition of a great glacier descending from Ben Nevis, and shutting up the valley of the Spean, *by resting on Moeldhu*, which is opposite, combined with the influence of a glacier from Loch Treig, and which would bar the valley a second time at that height, *would explain all the facts*."

These facts, for an explanation of which this theory was invented, appear to me not to have been accurately observed. In the first place, the *three* shelves do not occupy, as M. AGASSIZ asserts, " all the sinuosities of the lower part of Glen Spean, and of the whole of Glen Roy." It is only the lowest of the three shelves, which occurs in Glen Spean and in the lower part of Glen Roy. The two uppermost shelves stop short of the mouth of Glen Roy, by about 2 miles; so that, if the Lake in Glen Roy was dammed back by a terminal moraine, that moraine could not have rested on Moeldhu, at the foot of Glen Roy; but must have been pushed up that valley, before the Ben Nevis glacier, 2 miles farther,—an operation which the levels, distance, and direction of the valley would have rendered impossible.

In the second place, the shelves do not, as M. AGASSIZ says, " terminate at the same point,"—viz., at Moeldhu, where he supposes the terminal moraine of the Nevis glacier to have been. The two uppermost shelves (as just stated) do not come within two miles of this point; and the lowermost shelf, instead of *terminating* there, runs, as formerly explained, several miles northwards, on both sides of the valley, towards Unachan, where they are 4 miles apart. It is scarcely necessary to say, that a moraine in this low district, which is not connected with any Ben Nevis valley, and considering its required height and length, is inconceivable.

In the third place, as to the existence of transverse scratches on the rocks in Glen Spean, which are said to indicate the movement of some body from Loch Treig, I could see no such scratches, though I twice surveyed the ground, and narrowly inspected the rocks, especially at the outlet from Loch Treig. Indeed, the supposition that any glacier flowed out of Loch Treig seems to be almost excluded by the fact, that a shelf, perfectly horizontal, exists on both sides of the

narrow outlet from Loch Treig, and continuously into Glen Spean. Such a shelf could not have been formed, and would have been obliterated by any glacier moving out of Loch Treig.

(2.) But assuming all these facts to be as M. AGASSIZ states them, do they present unequivocal proofs of the movement of glaciers, and the formation of moraines? Scratches on polished rocks, may be made by various causes; and if a moraine existed on Moeldhu, surely some trace of it, or of the great blocks which generally accompany moraines, would have been particularly observable there;—whereas there is scarcely a block or a patch of gravel to be seen in that part of the valley.

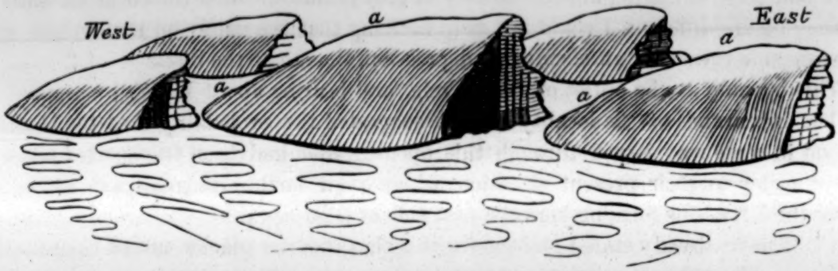
Farther, I would observe, that the valley supposed to have been the birthplace of the glacier, which produced this Moeldhu Moraine, is about two miles distant from Moeldhu, with an undulating country between them, which is most unlikely to have formed the channel or bed of a glacier. Dr BUCKLAND and M. AGASSIZ speak of this glen, as connected with Ben Nevis. But here, again, there is apparently some mistake. The valley in question is Larich Leachich, and runs up, not in a NW. direction towards Ben Nevis, but in a SW. direction towards the head of Loch Treig. It is an extremely short glen, and rises to no great height.

Finally, supposing, that if, in spite of all these objections, it were allowed that a glacier had moved down this little valley, and across the very uneven country to Moeldhu, so as to block up Glen Roy and Glen Spean, it would still remain to explain the blockage of Glen Gluoy, which, by no possibility, could be accounted for by a moraine at or near Moeldhu.

That there are certain appearances in the valleys of Lochaber, which must have been produced by attrition of some kind, I am free to admit. Water, accompanied by gravel and other detritus, appears, however, to have been the agent, and not ice. At the Monessie Falls, the valley is compressed to a narrow gorge, and the rocks forming the east side, present evident marks of attrition on a large scale, the rough faces of the rock being all down the valley. The rocks are here covered by sand and gravel, which indicate the flowing of water and of drift at that height, when these rocks were worn down. In like manner, at the outlet of Loch Treig, there are immense expanses of rock, all smoothed and rounded on the sides facing the SW. or WSW. by compass.* These smoothed rock-faces prevail to a height of about 786 feet above the lake, and 1680 feet above the sea, above which level they are no longer visible. There are many boulders lying on these smoothed surfaces, all of rounded forms. That these boulders have come from

* The general line or axis of the lake is north and south by compass, the upper part being towards the south, so that the motion of a glacier down this valley would have smoothed all the *south* faces of the rocks. It is also important to remark, that, on the west side of the lake, the rocks facing the lake are, as compared with those on the other side, exceedingly rough, shewing still more clearly that the smoothing agent had crossed the valley of Loch Treig, in a direction not parallel with its longer axis, but obliquely to it.

the west, is evident from the nature of them; several of a pink coloured felspar, having been traced by me to a dyke of the same peculiar rock a few hundred yards to the west, from which they had evidently been derived. Another circumstance proved this still more strikingly. In one place, a few hundred feet above Loch Treig, I observed a series of rocky knolls, in an east and west line. The parts of these knolls which were smoothed and worn down were uniformly to the west, whilst their rough faces were all to the east, thus—



It was clear, on an inspection of these knolls, that they had been worn down on their west sides; and the smoothed sides *a* were so close to the knolls respectively to the west of them, that nothing except some fluid, charged, it may have been, with drift, could have possibly reached and acted on them.

This last point was still more palpable, in several places, where there were narrow smooth-sided troughs, more or less steep, on the sides of hills. These troughs had apparently been natural fissures in the rocks, which had been smoothed by the long-continued action of water; for the notion that ice could have entered and rubbed them, was entirely precluded by their narrowness, situation, direction, and other circumstances.

M. Agassiz, in the paper before alluded to, says that he will never forget the impression he experienced " at the sight of the terraced mounds of blocks which occur at the mouth of the valley of Loch Treig, where it joins Glen Spean. It seemed to me (he adds) as if I were looking at the numerous moraines of the neighbourhood of Tines, in the valley of Chamounix." These terraces of blocks, thus likened to moraines, are, I presume, the accumulations of blocks on the lowermost horizontal shelf, which is very conspicuous at the entrance to Loch Treig on both sides of the valley. On this shelf there are multitudes of blocks, just as in many other parts of the valleys, where this shelf and the others occur. But this fact is perfectly consistent with the theory, that these shelves were formed by water, and, indeed, can be explained on no other, when it is considered that they form at Loch Treig, as at every other place, a line absolutely horizontal,—a quality which, I presume, no moraine ever possesses.

The only place where I observed an accumulation of blocks, at all resembling a moraine, is on the east side of Glen Spean, near a place called the Rough Burn.

about three or four miles to the north of Loch Laggan. The accumulation is enormous. Blocks are piled over each other, to such a height as to render the general surface of the moor, over a wide extent, quite undistinguishable. This accumulation occurs not at the mouth of any valley. On the contrary, the hills near these blocks on the east side, are not much furrowed even by mountain torrents, and present a somewhat steep and high wall face to the west. On looking round for any possible explanation of the occurrence in this spot of so unusual a quantity of boulders, consisting almost entirely of grey granites, whilst the rocks on which they lie are different, I could not help noticing that the valley on the opposite or west side presented an opening or depression, though at the distance of 2 miles. This opening is the outlet of Loch Treig, and bearing about WSW. by compass. The appearance of the locality at once suggested the probability that the blocks had in some way issued through this opening, and had been transported across the valley to their present situation, where their farther progress was arrested by the lofty hills forming here the east side of Glen Spean.

I have already stated reasons for thinking that no glacier issued from Loch Treig. The only alternative seems to be the agency of water.

I proceed now to shew that the lake theory of the Lochaber shelves, and the principles on which I have endeavoured to account for the formation of lakes, and the eventual depression and drainage of them, are not inconsistent with any established geological truths,—but, on the contrary, receive support from collateral considerations.

1. The first circumstance which I shall notice, *is the occurrence of Parallel Roads in other valleys similar to those of Lochaber*, the formation of which canbe attributed to no other cause than lakes.

I have the less hesitation in availing myself of this argument, when I find Mr Darwin adverting to traces of shelves at Kilfinnin, and in the valley of the Spey, in support of his theory.

But if Mr Darwin's views are sound, traces of shelves should not be confined to the two localities just mentioned; they should be visible in other parts of the country of equal height as the Lochaber mountains.

On the other hand, if it should appear that there are in many valleys, distinct beach lines, all horizontal, and presenting no uniformity of height above the sea, the argument against a sea theory will be strengthened, whilst a strong analogy will arise to favour the lake theory,—if these beach lines, precisely similar in all essential features to those of Lochaber, can, from their inland situation, and other circumstances, be clearly shewn to have been produced by the waters of lakes.

I proceed therefore to mention a few localities out of many, where phenomena similar to those of Glen Roy are observable.

(1.) At Inverournan (about 40 miles SW. of Lochaber) there is a lake called

Loch Tulla, about 3 miles in length, and 1 in breadth. A stream enters from its east and west ends. Its surplus waters are discharged from its south side, by the river Urchay.

Two years ago, I discovered all round this lake indications of three levels at which its waters had stood, the lowest being about 183½ feet, the second 277 feet, and the highest 474 feet, above their present level.* Loch Tulla I roughly estimated at 540 feet above the sea. This lake, therefore, extending originally to about 6 miles in length and half a mile in breadth, had sunk 197 feet,—at which level it had stood long enough to form the second shelf; it next sunk 93½ feet, —when the third shelf was formed; after which it sunk 183½ feet,—viz., to the present level of the lake.

It is unnecessary for me to enter into the proofs, that what I am now describing are really beach lines. Their perfect horizontality, which I ascertained by a spirit-level, looking at them from 12 or 15 different places along the banks of the lake,—their general conformity in sweeping round headlands, and retiring into valleys or burn-courses,—and the extent of flat surface at the levels of the different shelves, afford convincing and irrefragable proofs.

The difficulty here, as in other similar cases, is to discover, what could have dammed up the lake so much above its present level. The blockage, whatever it was, must have existed somewhere in the valley, through which the river Urchay flows. The country, on all other sides of Loch Tulla, rises much higher than 500 feet above its present level. The two lowest shelves are traceable for some distance down the valley of the Urchay,—the middle shelf for about half a mile, and the lowest considerably farther. My notion is, that this valley had been formerly filled with a great accumulation of gravel and diluvial debris, which was gradually eat away and lowered by the stream which issued from the loch. Accordingly, there exist still, at and near Urchay Bridge, great heaps of unstratified gravel, which clearly present only a remnant of what must have formerly existed. The valley at this place, is a quarter of a mile wide; and its sides rise far above the required level.

(2.) In the valley, at the head of which Tyndrum is situated, there are very manifest indications of the beaches of an ancient lake, although the valley is now occupied by only an insignificant stream. At Strathfillan church, the lowest terrace is about 50 or 60 feet above the stream, and may be traced continuously for at least a mile down the valley. The stream has cut through this old lake bottom, exhibiting beds of gravel, sand, and clay, which have been deposited and arranged by the water. About 237 feet above this flat, there are, on the sides of the hills on both sides of the valley, traces of a horizontal shelf, which can be distinctly followed with the spirit-level from above Tyndrum village, down the valley by Auchreach farm-houses, Enich farm-houses, and as far as Crianlarich

* These measurements were made by a mountain barometer, checked by the sympiesometer.

toll. At several places, boulders appear to have accumulated on this higher shelf. Tyndrum is about 740 feet above the sea.

(3.) Along the margin of Loch Awe, and particularly near Dalmally, there is a flat or terrace about 40 feet above the present level of the lake; and which manifestly indicates a subsidence of its waters to that depth.

(4.) Along the margin of Loch Lubnaig, in like manner, there is a flat or terrace about 40 feet above the lake, and which is very visible on both sides. Here as well as in the former case, the flat runs back from near the margin of the lake to the mountains forming one side of the valley; and the steep sides of which, contrast most significantly with the almost horizontal flatness of the ancient and exposed bottom.

At Loch Lubnaig, the flat can be traced for a considerable way on both sides of the valley, beyond the point where the lake now discharges itself, and, indeed, almost as far as Leny. At this place, as well as at Callendar, there exist indications of enormous quantities of gravel, which, before being cut down and carried away by rivers, afforded ample means of blocking up the waters of Loch Lubnaig to a higher level. The quantity of gravel which formerly existed hereabouts, may be inferred from the existence of the following remnants.

About $\frac{1}{4}$ mile west of Callendar, there is a ridge of gravel and sand about 100 yards long, and from 40 to 50 feet high. Near it, there is a conical mound of the same materials, and about the same height, bearing a thriving plantation. The ridge of gravel to the east of Callendar, designated in guide-books as the Roman Camp, is merely a remnant of the ancient gravel-bed with which the whole valley was filled; and when it contained a lake, of which there are abundant indications, it is probable, that, when Loch Lubnaig stood 40 feet above its present level, its waters were discharged into a lower lake, of which the eastern margin may be seen near the Lodge of Gart-House. Ultimately the gravel heaps which held in this Callendar lake on the east, had been cut through, so as to allow of its drainage; and, accordingly, there are, on each side of the river Teith at this place, gravel banks and cliffs from 70 to 80 feet high.

After the Callendar lake was drained, the waters which flowed out of Loch Lubnaig would acquire fresh cutting power, and would rapidly eat away the barrier which dammed back the lake to the higher level before referred to.

Callendar is about 270 feet above the sea.

(5.) In the valley in which the town of Huntly stands, there are two terraces, the one about 32 feet above the other, which are very clearly the beaches of a lake, which has sunk from the one to the other, and latterly been drained off.

(6.) A few miles south of Inverury, there are distinct traces of a lake which formerly filled the valley. The burgh of Kintore has been built in the ancient bottom of the lake. There are two well-marked beach-lines round the whole valley;

the one about 78 feet, and the other 50 feet, above the channel of the united streams of Don and Urie, which flow through the centre of the valley. The ancient bottom of the lake has been cut up by rivulets at the sides of the valley into separate fragments, some of them of so unusual a form as to have suggested a notion that they are artificial; and, accordingly, in the guide-books, and even in the recent statistical accounts of the parish, they are so described. Two of these alleged remains of antiquity are known by the names of Bass and Konin Hillock; and are variously conjectured to have been formed for sepulchral or judicial purposes. A similar mistake has been made with the hills of Dunipace, near Falkirk, which are represented by historians as formed to celebrate and record a peace between the Romans and the natives of Scotland. They are detrital remnants fashioned into conical shapes by the action of streams.

(7.) In the valley of the Leader (Berwickshire), there will be found terraces on the hill sides, which clearly shew the action of water. Three very distinct markings of this nature are traceable near Dodds' Mill, at Hounslow, at Carfrae Mill, and at Annfield near Channelkirk. The terraces at these different places, judging by the sympiesometer, seem to be all very nearly on a level; and if, on a more minute survey, they really prove to be so, it would follow, that the whole of Lauderdale had formerly been one vast lake, with a blockage at or near Chappel. The height of these shelves is about 800 feet above the sea.

It is scarcely necessary to advert to the inland situation, and other circumstances characteristic of the various beach-lines now mentioned, to shew that they could not have been formed by the sea, but must have been produced by lakes which filled the valleys, and which sunk at different periods,—in most cases, disappearing altogether.

If, then, the existence of lake-beaches be so common in the valleys of Scotland, there will be the less hesitation in ascribing the Lochaber shelves to the same cause,—established as that cause has been separately by local evidence.

That the occurrence of lake-beaches in the valleys of Scotland should be frequent, is only what every geologist must be prepared to expect, who considers the proofs which may be adduced, of the gradual emergence of the land out of the sea. Some of these proofs, in so far as afforded by Scotland, I shall immediately notice; but assuming that Scotland was, to the depth of 1300 feet or more, submerged beneath the waters of the ocean,—as it rose out, there would be lakes in every inland hollow, each, of course, having its river to carry off to the sea, the rain falling on its surface and that of the adjoining mountains. The stream thus issuing, would gradually wear down the detritus which formed a barrier at one end of the lake; and the cutting power of the stream would be gradually increased, as the elevation of the land proceeded; so that in most cases the blockage of lakes would, in the course of time, be extensively undermined

and worn down, and sudden depressions of lakes would take place, leaving marks of horizontal shelves along the sides of valleys.

The progress of these important changes is indicated, in many parts of the country, by the existence of haughs or river-flats, far above the present channels of the streams, and which evidently had been formed when they flowed at a much higher level.

Thus, from Perth up to Loch Tay, a number of isolated flats or terraces occur, forming a pretty uniform level, rising gently inland, and at a rate rather faster than the slope of the river. Near Perth, these old haughs are from 90 to 100 feet, and at Dunkeld about 110 feet, above the river. This old haugh at Dunkeld may be traced on both sides of the valley,—Dr Fisher's house being on it at the east side, and Claypotts farm-house on it at the west side. It may even be traced a considerable distance up both sides of the Braan, where it slopes a little to the eastward.

There is a low haugh at Dunkeld which is only about 20 feet above the present bed of the river, and is, therefore, quite distinct from the higher terrace above described. The ground is now cultivated and enclosed; so I suppose that the floods never rise to a level with it now.

On the Tweed, in like manner, the remains of ancient haughs can be traced in many part of its course. About half a mile above Berwick Bridge, one may be seen on the south side, from 30 to 32 feet above the sea. At Gainslaw, it is 44 feet; opposite to Finchie, it is 55 or 56 feet; opposite to Paxton, it is 58 feet; at Norham, it is 93 feet above the sea.

At New Rattray (in the parish of Blairgowrie) I observed an extensive flat, or ancient haugh, with its cliff or bank about 80 feet above the River Ericht.

On the Isla, above Airley Castle, there is haugh land, on both sides, about 30 feet above the present level of the river.

On the River Garry, about 3½ miles north of Blair, there are on the east side two terraces, the one about 30 and the other about 50 feet above the river; but whether they are the remains of ancient haughs, or the beaches of a lake, it is difficult to determine.